Math 131/135/194, Fall 2004

Applied Calculus

Profs. Kaplan & Flath

Macalester College

Assignment: Linear Algebra

Problem 1

1

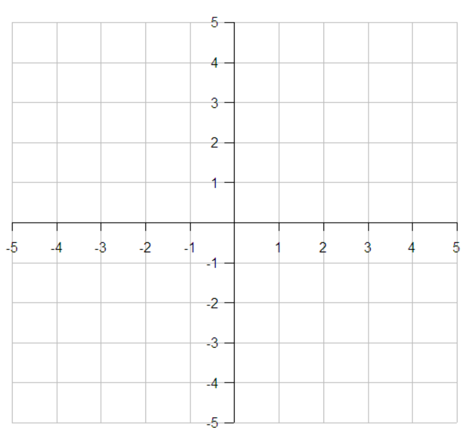
Find the angle between the following pairs of vectors, drawing the vectors on the graph paper to confirm that your answer is right. (For your convenience, a graph of the cosine function is given)

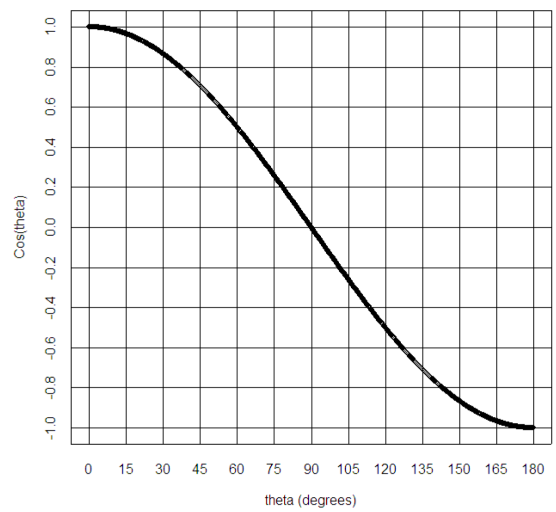
1. $(1, 0)$ and $(0, 1)$

2. $(1, 1)$ and $(1, 0)$

3. $(1, 1)$ and $(-1, 1)$

4. $(3, 2)$ and $(-3, -2)$





Problem 2

2

Find the angles between the following pairs of vectors:

1. $(5, 2, 3)$ and $(-3, 2, -3)$

2. $(7, 4, 2, 3)$ and $(4, 2, -3, 7)$

Problem 3

6

Calculating the mean (average) of the components in a vector is equivalent to projecting the vector onto the vector of all ones: $(1, 1, \cdots)$.

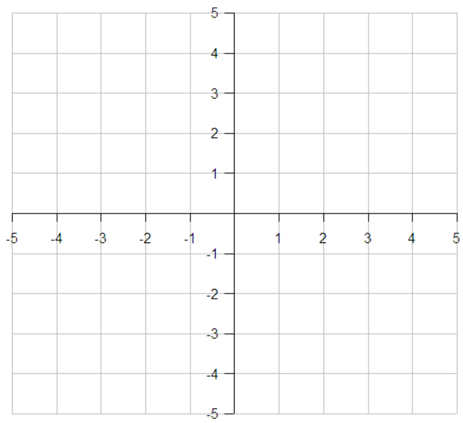
Project onto the vector of ones each of the following vectors:

1. $(4, 2)$

2. $(-3, 3)$

3. $(5, 5)$

4. $(7, 4, -3, 4)$



Problem 4

3

In statistics, the *correlation coefficient* is a way of describing the relationship between two variables. When the correlation coefficient is 1, any change in one variable is perfectly reflected in the other. When the correlation coefficient is 0, a change in one variable is not (on average) reflected in the other.

For a vector $\mathbf{u}$, the vector $\bar{\mathbf{u}}$ is the vector all of whose components equal the average of the components of $\mathbf{u}$. For example , for $\mathbf{u} = (3, 5, 6, 2)$ we have $\bar{\mathbf{u}} = (4, 4, 4, 4)$.

As it happens, the correlation coefficient between two vectors $\mathbf{u}$ and $\mathbf{v}$ is the cosine the cosine of the angle between the two vectors $\mathbf{u} - \bar{\mathbf{u}}$ and $\mathbf{v} - \bar{\mathbf{v}}$.

Calculate the correlation coefficient between these vectors: $\mathbf{v} = (3, -4, 5, 2, 3, -6, 8)$ and $\mathbf{w} = (9, 2, 3, 5, 1, -2, 4)$.

Problem 5

4

Find three differently pointing vectors orthogonal to $(3, 2, 5, 4)$.

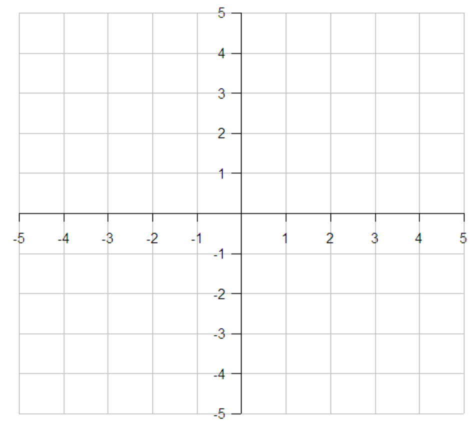
Problem 6

5

Using graph paper, project the vector $(3, 2)$ onto the target $(1, 0)$.

1. Give numerically both the projected vector and the residual vector.

2. Show numerically that the residual is orthogonal to the target vector.



Problem 7

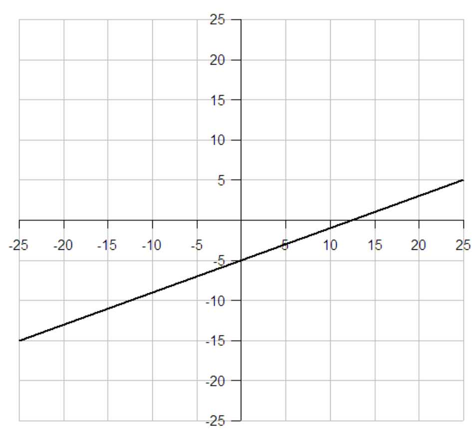
7

Is the vector $(3, 1, 2)$ orthogonal to the plane spanned by $(0, 2, -1)$ and $(-1, 3, 0)$?

Problem 8

8

For the line shown in the graph below, give a description of the line in terms of a vector sum and a scalar parameter.



Problem 9

9

Imagine the surface of a table in a room where a coordinate system has been established with origin in the corner of the room and axes running along the joints between the floor and each of the 2 walls, and between the two wall themselves. Give a description of the 2-dimensional space in which the table surface lies in terms of a two vectors in terms of the formula: $\mathbf{v} + \alpha \mathbf{w}$. Specify $\mathbf{v}$ and $\mathbf{w}$.

Problem 10

10

Consider two vectors, $\mathbf{p} = (1, 2)$ and $\mathbf{q} = (3, -2)$.

1. For several values of the scalars α and β, that you pick, plot out the position of $\alpha\mathbf{p} + \beta\mathbf{q}$. Explain in words what points in the space can be reached by appropriate choice of α and β.
2. Repeat the above, but holding $\alpha = 1$ and setting β however you want. What points in the space can be reached this way?
3. Now hold $\beta = 1$ and setting α however you want. What points in the space can be reached this way?
4. Now pick β as you want, but set $\alpha = 1 - \beta$, so that $\alpha + \beta$ always equals 1. What points in the space can be reached this way?

Problem 11

11

Consider the function

$$f(\mathbf{x}) = x_1^2 + x_2^3 + x_3 x_1$$

Find 2 distinct vectors, which we'll call $\mathbf{v}_a$ and $\mathbf{v}_b$, which are orthogonal to the gradient at $(1, 3, 1)$.

Explain the geometry of the set of points that can be reached by the vector sum $(1, 3, 1) + \alpha\mathbf{v}_a + \beta\mathbf{v}_b$, where α and β are scalars.

Problem 12

12

We often measure variables which are *categorical* rather than numerical. For example, when studying trees, we might record the species of the tree as one of several possibilities: oak, maple, birch, etc. Sexes are recorded as either male or female. In this problem, we'll imagine that the possible categories are A, B, C, or D. The possible values are called the *levels* of the variable.

Suppose that we measure 10 cases, giving us the categories (A,A,B,D,D,A,D,B,B,C). An indicator variable for a given level is a numerical vector that has the value 1 for those cases that match the level, and 0 for the other cases. For example, the indicator variable for level A would be (1,1,0,0,0,1,0,0,0,0).

When there are 4 levels, there will be 4 distinct indicator variables. Show that all 4 of these indicator variables are mutually orthogonal.

1. Explain why, for any categorical variable, the set of indicator variables must always be mutually orthogonal.
2. Show that the vector $(1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ can be written as a linear combination of the indicator variables for A, B, C, and D.
3. Explain why, for any categorical variable, the vector of all 1s can always be written as a linear combination of the indicator variables drawn from that variable.

Problem 13

13

On the paper, show how $\mathbf{w} = (1, -8)$ is a linear combination of $\mathbf{u} = (4, 1)$ and $\mathbf{v} = (-3, 2)$.