The Twenty-First Annual Konhauser Problemfest

Carleton College, February 23, 2013

Problems set by George Gilbert, Texas Christian University

This contest is held annually in memory of Professor Joseph Konhauser (1924-1992) of Macalester College, who posted nearly 700 Problems of the Week at Macalester over a 25-year period. Joe died in February of 1992, and the contest was started the following year.

SUMMARY OF INSTRUCTIONS: (You will have received somewhat more detailed instructions before the contest.) Each team must hand in all work to be graded at the same time (after three hours). Each problem must be written ON A SEPARATE PAGE (or pages) and YOUR TEAM NAME MUST BE WRITTEN AT THE TOP OF EVERY PAGE. Only one version of each problem will be graded for each team, so do not submit multiple versions. Calculators of any sort are allowed, but answers should be exact, and justifications and/or explanations are expected for all problems. All ten problems will be weighted equally; partial credit will be given for substantial progress toward a solution. Solutions that are especially difficult to follow or hard to read may not get full credit, even if they turn out to be correct. Of course, you may not get information about anything related to these problems from anyone or anything except:

a) You are encouraged to work with your team members and

b) If a problem seems incorrect or unclear, you may ask supervising faculty.

1. An Explicit Fair Division

Consider the points $A = (4, 0)$, $B = (0, 3)$, $C = (0, 0)$ in the plane. Find all pairs of points P and Q on $\triangle ABC$ that divide the perimeter of the triangle in half and such that PQ divides the area of the triangle in half.

2. Is That All There Is?

Let S be a nonempty set of positive integers. Suppose that for every positive integer n, if any one of the three positive integers $n, 2n + 9,$ and $2n + 25$ is in S, then all three are in S. Does it follow that S is the set of all positive integers?

3. A Creepy, Decreasing Sequence

Let a_0 be a positive rational number. For every integer $n \geq 0$, let k_n be the smallest positive integer for which $a_n - \dfrac{1}{k_n} > 0$, and define $a_{n+1} = a_n - \dfrac{1}{k_n}$. Must $\dfrac{1}{a_n}$ be an integer for infinitely many n ?

4. Missing the First Derivative

For what real numbers a other than 0 and 1 does there exist a polynomial $p(x)$ (with real coefficients) of degree 5 such that

$$p(0) = p(1) = p(a) = 0\,,$$
$$p''(0) = p''(1) = p''(a) = 0\,?$$

5. Losing One's Marbles?

Players A and B play a game in which they take turns removing marbles from a bowl that begins with n marbles. On his turn, player A must remove 1 or 2 marbles; on her turn, player B must remove $2011, 2012,$ or 2013 marbles. The winner is the last player who is able to remove (a legal number of) marbles. Assuming that both players play optimally, which of the following three sets is/are infinite?

$$S_A = \{n|\text{ Player A will win regardless of who moves first}\}$$

$$S_B = \{n|\text{ Player B will win regardless of who moves first}\}$$

$$S_C = \{n|\text{ The winner depends on who moves first}\}$$

6. Seeding Integer Roots

Let $p_0(x) = x^2 + ax + b$ and, for all positive integers n, let $p_n(x) = p_{n-1}(x) + x + 2$. Find all pairs of integers (a, b) such that the roots of $p_n(x)$ are integers for every integer $n \geq 0$.

7. Two Possible Endings

A standard fair die (with six faces numbered $1, 2, 3, 4, 5, 6$) is rolled until either a 6 appears or two consecutive 1's appear. Find the probability that the process stops after exactly n rolls. (Express your answer in closed form.)

8. Does Dimension Really Matter?

Let G_n denote the set of all invertible $n \times n$ matrices with real entries. Is there a bijection (one-to-one and onto function) $f : G_2 \to G_3$ such that $f(AB) = f(A)f(B)$ for all A and B in G_2?

9. A Fine Line Between Convergence and Divergence

Let $\left(a_n\right)_{n\geq 1}$ be a sequence of positive numbers that have limit 0. Must there exist a sequence (b_n) of positive numbers such that $\sum_{n=1}^{\infty} b_n$ diverges and $\sum_{n=1}^{\infty} a_n b_n$ converges?

10. Pushing the e**nv**e**lop**e

Let d be a positive integer, and let S_d be the set of all the polynomials $p(x)$ of degree at most d such that all of $p(0), p'(0), p''(0), \ldots$ are nonnegative integers. Prove that there exists a polynomial $q_d(x)$ in S_d such that the following are equivalent for all $p(x)$ in S_d:

(i) $p(1/n) \leq q_d(1/n)$ for all positive integers n;

(ii) $p(1/n)^n < e$ for all positive integers n.

The Twenty-First Annual Konhauser Problemfest - Solutions

Carleton College, February 23, 2013

1. An Explicit Fair Division

Consider the points $A = (4, 0)$, $B = (0, 3)$, $C = (0, 0)$ in the plane. Find all pairs of points P and Q on $\triangle ABC$ that divide the perimeter of the triangle in half and such that PQ divides the area of the triangle in half.

Solution. We'll see that there is only one such pair of points P, Q (up to order). Note that neither of the points can be at a vertex, because then the other point would have to be at the midpoint of the opposite side (for the area to be divided equally), and then the perimeter isn't divided equally. Let the lengths of the sides that P and Q are on be x and y, and let the distances from P and Q to the vertex V (one of A, B, C) where those sides meet be p and q, respectively. From the condition on the perimeter, we get $p + q = 6$. The area of a triangle is half the product of two sides and the sine of their included angle, so the area of the whole triangle is $(xy \sin\theta)/2$, where θ is the angle at V. The condition on the area gives us $(pq \sin\theta)/2 = (xy \sin\theta)/4$, so $2pq = xy$.

Eliminating q yields

$$2p^2 - 12p + xy = 0,$$

with solutions

$$p = 3 \pm \sqrt{9 - xy/2}.$$

Thus, there is no loss of generality in assuming

$$p = 3 + \sqrt{9 - xy/2}, \quad q = 3 - \sqrt{9 - xy/2}.$$

If $\{x, y\} = \{4, 5\}$, then $9 - xy/2 < 0$, an impossibility. If $\{x, y\} = \{3, 4\}$, then $p = 3 + \sqrt{3} > 4$, also an impossibility. Thus, $\{x, y\} = \{3, 5\}$, $p = 3 + \sqrt{3/2}$, and $q = 3 - \sqrt{3/2}$. Now we know that the vertex V is B, and that P is on BA (because $p > 3$) while Q is on BC. Then we see from the distances p and q that $Q = (0, \sqrt{3/2})$, while

$$P = (0, 3) + (3 + \sqrt{3/2})\,(4/5, -3/5) = \left(\frac{12 + 2\sqrt{6}}{5}, \frac{12 - 3\sqrt{6}}{10}\right).$$

Note. This solution is more efficient than setting up separate calculations for the three possible pairs of sides that P and Q might lie on, but of course that can be done also.

2. Is That All There Is?

Let S be a nonempty set of positive integers. Suppose that for any positive integer n, if any one of the three positive integers $n, 2n + 9,$ and $2n + 25$ is in S, then all three are in S. Does it follow that S is the set of all positive integers?

Solution 1. Yes. Because S is nonempty, it is enough to show that n is in S if and only if $n + 1$ is in S. Define $f(x) = 2x + 9$. Then $f(k) - f(j) = 2(k - j)$, so that for the three-fold composition $f^{(3)}$, we have $f^{(3)}(n+1)-f^{(3)}(n) = 8$. Therefore, $2\cdot f^{(3)}(n+1)+9 = 2\cdot f^{(3)}(n)+25$. If this integer is in S, so are n and $n+1$. If not, neither is in S.

Solution 2. Because

$$2(2n+9)+9 = 2(2n+1)+25 \qquad (*),$$

n is in S iff (if and only if) $2n+9$ is in S, iff both sides of $(*)$ are in S, iff $2n+1$ is in S. Then because $2n+9 = 2(n+4)+1$, n is in S iff $n+4$ is in S. Then because $(2n+9)+4 = 2(n+2)+9$, n is in S iff $n+2$ is in S. Finally, because $(2n+9)+2 = 2(n+1)+9$, n is in S iff $n+1$ is in S, and because S is nonempty, we are done.

Note. The problem wouldn't really change if we replaced "positive integer(s)" by "integer(s)" throughout.

3. A Creepy, Decreasing Sequence

Let a_0 be a positive rational number. For every integer $n \geq 0$, let k_n be the smallest positive integer for which $a_n - \dfrac{1}{k_n} > 0$, and define $a_{n+1} = a_n - \dfrac{1}{k_n}$. Must $\dfrac{1}{a_n}$ be an integer for infinitely many n ?

Solution. Yes; in fact, once $1/a_n$ is an integer, so are all $1/a_m$ for $m > n$.

To begin, note that if $a_0 > 1$, then $k_0 = 1, a_1 = a_0 - 1$, and we keep subtracting 1 from a_n until $0 < a_n \leq 1$. This takes finitely many steps, so we might as well assume $0 < a_0 \leq 1$. Now if $1/a_0$ happens to be an integer m , then $k_0 = m+1$, and

$$a_1 = \frac{1}{m} - \frac{1}{m+1} = \frac{1}{m(m+1)}$$

is again the reciprocal of an integer, as claimed. Otherwise, write a_0 in lowest terms as c_0/d_0 ; we know that $1 < c_0 < d_0$. Divide d_0 by c_0 with remainder: $d_0 = qc_0 + r$; we know that the remainder is positive (else $a_0 = 1/q$), and because $r < c_0$ we have $qc_0 < d_0 < (q+1)c_0$, hence $1/(q+1) < a_0 < 1/q$ and so $k_0 = q+1$. Then

$$a_1 = a_0 - \frac{1}{q+1} = \frac{c_0 - r}{d_0(q+1)} \, .$$

Thus a_1 is again a positive rational between 0 and 1, with a smaller numerator than a_0. Because the numerator must keep decreasing, it will reach 1 in finitely many steps, and we are done.

4. Missing the First Derivative

For what real numbers a other than 0 and 1 does there exist a polynomial $p(x)$ (with real coefficients) of degree 5 such that

$$p(0) = p(1) = p(a) = 0 \, ,$$

$$p''(0) = p''(1) = p''(a) = 0 \, ?$$

Solution. For $a = -1, 1/2, 2$. Because $p(x)$ has degree 5, $p''(x)$ has degree 3, so we know all its roots are $0, 1,$ and a. After multiplying $p(x)$ by a constant, we may assume that

$$p''(x) = x(x-1)(x-a) = x^3 - (a+1)x^2 + ax\,.$$

Integrating twice,

$$p(x) = \frac{1}{20}x^5 - \frac{a+1}{12}x^4 + \frac{a}{6}x^3 + Cx + D$$

for some constants C and D. From $p(0) = 0$, we get $D = 0$. From $p(1) = 0$, we get $1/20 - (a+1)/12 + a/6 + C = 0$ or $C = -a/12 + 1/30$. Finally, $p(a) = 0$ yields

$$\frac{a^5}{20} - \frac{(a+1)a^4}{12} + \frac{a^4}{6} + Ca = 0\,.$$

Substituting in our expression for C, dividing by $-a$, and simplifying leads to

$$\frac{a^4}{30} - \frac{a^3}{12} + \frac{a}{12} - \frac{1}{30} = 0\,.$$

We see that there is a factor $a^2 - 1$, and we then get

$$\frac{1}{60}(a^2-1)(2a-1)(a-2) = 0\,.$$

Discarding the root $a = 1$, we see that a can be $-1, 1/2,$ or 2, as claimed.

Note. The values for a are precisely the three values for which $0, 1,$ and a, in some order, are equally spaced on the number line. It would be interesting to have a solution that shows directly that this is necessary and sufficient for the existence of $p(x)$.

5. Losing One's Marbles?

Players A and B play a game in which they take turns removing marbles from a bowl that begins with n marbles. On his turn, player A must remove 1 or 2 marbles; on her turn, player B must remove $2011, 2012,$ or 2013 marbles. The winner is the last player who is able to remove (a legal number of) marbles. Assuming that both players play optimally, which of the following three sets is/are infinite?

$$S_A = \{n \mid \text{ Player A will win regardless of who moves first}\}$$
$$S_B = \{n \mid \text{ Player B will win regardless of who moves first}\}$$
$$S_C = \{n \mid \text{ The winner depends on who moves first}\}$$

Solution 1. We'll show that only S_B is infinite. First note that 2013 is in S_B, because if B moves first she can take all the marbles, while if A moves first and takes i marbles, B wins by taking $2013 - i$ marbles. Similarly, if n is in S_B, then $n + 2013$ is in S_B, because B can always arrange to get from $n + 2013$ to n marbles in either one or two moves. Therefore, S_B contains all positive multiples of 2013 and is thus infinite.

Now note that if n is in S_B, then B wins if there are $n+1$ marbles and she moves second, because if A takes one marble B has a winning position, and if A takes two marbles it will be as if he started with n marbles and took one; by the assumption that n is in S_B, B has a strategy to deal with that move and win. On the other hand, if B wins when there are k marbles and she moves second, then $k+2013$ is in S_B, because B can either take 2013 marbles if it is her move, or respond to A taking i marbles by taking $2013-i$. In particular, for $k=n+1$, this shows that if n is in S_B, then so is $n+2014$; note that $n+2014 = n+1 \pmod{2013}$. Because we know that all multiples of 2013 are in S_B, we now see by induction that all sufficiently large numbers that are congruent to any of the possible remainders $1, 2, \ldots \pmod{2013}$ are in S_B, and we are done.

Solution 2. (Izabella Łaba). Because 2013 and 2014 are relatively prime, every sufficiently large n can be written in the form $n = 2013j + 2014k$ where j and k are positive integers. Suppose it is A's move with such a number n of marbles in the bowl. Here is a winning strategy for B: In the first k rounds, if A chooses i, B responds by choosing $2014-i$. In the following j rounds, if A chooses i, B chooses $2013-i$. Thus, B wins if A moves first. On the other hand, if B moves first she can still win by first taking 2011 marbles and then following the same strategy with n replaced by $n-2011$ to respond to A's moves, as long as $n-2011$ is large enough. Thus for sufficiently large n, B will win regardless of who moves first, and so only S_B is infinite.

6. Seeding Integer Roots

Let $p_0(x) = x^2 + ax + b$ and, for all positive integers n, let $p_n(x) = p_{n-1}(x) + x + 2$. Find all pairs of integers (a, b) such that the roots of $p_n(x)$ are integers for every integer $n \geq 0$.

Solution 1. By direct computation,

$$p_n(x) = x^2 + (n+a)x + (2n+b) \quad \text{has roots} \quad x_{1,2} = \frac{-n-a \pm \sqrt{(n+a)^2 - 4(2n+b)}}{2}.$$

The discriminant $(n+a)^2 - 4(2n+b)$ has the same parity (even or odd) as $-n-a$, so these roots are integers if and only if the discriminant is a perfect square, and we have to find the (a,b) such that this is true for all n. Now

$$\begin{aligned}(n+a)^2 - 4(2n+b) = n^2 + (2a-8)n + (a^2-4b) &= (n+a-4)^2 - (a-4)^2 + (a^2-4b) \\ &= (n+a-4)^2 + (8a-4b-16).\end{aligned}$$

This is clearly a perfect square (for any n) if $8a-4b-16=0$. On the other hand, if $8a-4b-16 \neq 0$, then for large enough n the absolute value of the difference between $(n+a-4)^2$ and any other perfect square will be greater than $|8a-4b-16|$, and then the discriminant cannot be a square for such n. So the desired pairs (a,b) are the ones for which $8a-4b-16=0$, that is, $b = 2a-4$.

Solution 2. We'll show that the set consists of all pairs of integers for which $2a-b=4$. Observe that if $p_n(x) = x^2 + a_n x + b_n$, then

$$2a_{n+1} - b_{n+1} = 2(a_n+1) - (b_n+2) = 2a_n - b_n.$$

Thus $2a_n - b_n$ is constant for the sequence of polynomials. It follows that if $2a - b = 4$, then each $p_n(x)$ has the form $x^2 + a_n x + (2a_n - 4)$; this polynomial has roots -2 and $2 - a_n$, which are integers, as desired.

We now show that the condition $2a - b = 4$ is also necessary. For $p_0(x)$ and $p_1(x)$ to have integral roots, their discriminants, $a^2 - 4b$ and $(a+1)^2 - 4(b+2)$, must be perfect squares. Either a or $a+1$ is odd, so the corresponding discriminant is odd and $4b$ or $4(b+2)$ is the difference of odd squares, hence a multiple of 8. Therefore, b is even. For some n, $b + 2n = 2p$ for some odd prime p. There is no loss of generality in assuming $b = 2p$, $p_0(x) = x^2 + ax + 2p$. Then the discriminant $a^2 - 8p$ is a perfect square which cannot be zero, so we have $a^2 - 8p = d^2$ for some positive integer d, which yields $(a-d)(a+d) = 8p$. Because a and d have the same parity, either (i) $a - d = 4$, $a + d = 2p$ or (ii) $a - d = 2$, $a + d = 4p$. In case (i), $a = p + 2$, $2a - b = 4$, and we are done. In case (ii), $a = 2p + 1$. But then $p_1(x)$ has discriminant

$$(2p+2)^2 - 4(2p+2) = 4p^2 - 4.$$

However, the only perfect squares differing by 4 are 0 and 4, which would imply $p = 1$, a contradiction.

7. Two Possible Endings

A standard fair die (with six faces numbered $1, 2, 3, 4, 5, 6$) is rolled until either a 6 appears or two consecutive 1's appear. Find the probability that the process stops after exactly n rolls. (Express your answer in closed form.)

Solution 1. The probability, for $n \geq 1$, is

$$\frac{1}{4}\left(\frac{1+\sqrt{2}}{3}\right)^n + \frac{1}{4}\left(\frac{1-\sqrt{2}}{3}\right)^n.$$

Let p_n denote the probability that the process stops on or before n rolls and let q_n denote the probability that the process has not stopped after n rolls and the nth roll is a 1. We have $p_0 = q_0 = 0$ and $p_1 = q_1 = 1/6$. For $n \geq 1$,

$$\begin{aligned} p_n &= p_{n-1} + \frac{1}{6}(1 - p_{n-1}) + \frac{1}{6}\, q_{n-1} = \frac{1}{6} + \frac{5}{6}\, p_{n-1} + \frac{1}{6}\, q_{n-1}\,, \\ q_n &= \frac{1}{6}(1 - p_{n-1} - q_{n-1})\,. \end{aligned}$$

Thus

$$\begin{aligned} p_{n+1} + \frac{1}{6}\, p_n &= \frac{7}{36} + \frac{5}{6}\, p_n + \frac{5}{36}\, p_{n-1} + \frac{1}{6}\left(q_n + \frac{1}{6}\, q_{n-1}\right) \\ &= \frac{7}{36} + \frac{5}{6}\, p_n + \frac{5}{36}\, p_{n-1} + \frac{1}{36}(1 - p_{n-1}) \\ &= \frac{2}{9} + \frac{5}{6}\, p_n + \frac{1}{9}\, p_{n-1}\,, \end{aligned}$$

or

$$9p_{n+1} - 6p_n - p_{n-1} = 2\,.$$

The characteristic equation for this difference equation has roots $(1 \pm \sqrt{2})/3$, so the homogenous solution is

$$A_+ \left(\frac{1+\sqrt{2}}{3}\right)^n + A_- \left(\frac{1-\sqrt{2}}{3}\right)^n .$$

Noting that 1 is a particular solution, we have

$$p_n = 1 + A_+ \left(\frac{1+\sqrt{2}}{3}\right)^n + A_- \left(\frac{1-\sqrt{2}}{3}\right)^n .$$

From p_0 and p_1, we find

$$A_+ = \frac{-4-3\sqrt{2}}{8}, \qquad A_- = \frac{-4+3\sqrt{2}}{8}.$$

The probability of stopping on the nth roll is $p_n - p_{n-1}$, which simplifies to

$$p_n - p_{n-1} = \frac{1}{4}\left(\frac{1+\sqrt{2}}{3}\right)^n + \frac{1}{4}\left(\frac{1-\sqrt{2}}{3}\right)^n .$$

Solution 2. There are only three "states" that the process can be in after any number of rolls: It can have just stopped, it can be in the "starting" state (if the previous roll was not a 1 or a 6), or it can be in the "one" state, where the previous roll was a 1. Let the probabilities that the process stops after exactly n more rolls be P_n for the starting state (this is what we want to compute) and Q_n for the "one" state. Note that $P_1 = 1/6$ and $Q_1 = 2/6 = 1/3$ (in the "one" state, rolling either a 1 or a 6 stops the process). Also, when we roll once from the starting state we get back to the starting state with probability 2/3 and we get to the "one" state with probability 1/6. Therefore, for $n > 1$,

$$P_n = \frac{2}{3}P_{n-1} + \frac{1}{6}Q_{n-1}\,,$$

and similarly,

$$Q_n = \frac{2}{3}P_{n-1}\,.$$

We can rewrite this pair of recurrence relations as the matrix equation

$$\begin{bmatrix} P_n \\ Q_n \end{bmatrix} = \begin{bmatrix} 2/3 & 1/6 \\ 2/3 & 0 \end{bmatrix} \begin{bmatrix} P_{n-1} \\ Q_{n-1} \end{bmatrix},$$

and using the initial conditions we conclude that

$$\begin{bmatrix} P_n \\ Q_n \end{bmatrix} = \begin{bmatrix} 2/3 & 1/6 \\ 2/3 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 1/6 \\ 1/3 \end{bmatrix} = \frac{1}{6^n}\begin{bmatrix} 4 & 1 \\ 4 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

To compute the matrix power, we can diagonalize. By direct computation, the eigenvalues of $\begin{bmatrix} 4 & 1 \\ 4 & 0 \end{bmatrix}$ are $\lambda = 2 \pm 2\sqrt{2}$, with corresponding eigenvectors $\begin{bmatrix} 1 \\ \pm 2\sqrt{2} - 2 \end{bmatrix}$, so we have

$$\begin{bmatrix} 4 & 1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2\sqrt{2}-2 & -2\sqrt{2}-2 \end{bmatrix} \begin{bmatrix} 2+2\sqrt{2} & 0 \\ 0 & 2-2\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2\sqrt{2}-2 & -2\sqrt{2}-2 \end{bmatrix}^{-1} \quad \text{and}$$

$$\begin{bmatrix} 4 & 1 \\ 4 & 0 \end{bmatrix}^{n-1} = \begin{bmatrix} 1 & 1 \\ 2\sqrt{2}-2 & -2\sqrt{2}-2 \end{bmatrix} \begin{bmatrix} (2+2\sqrt{2})^{n-1} & 0 \\ 0 & (2-2\sqrt{2})^{n-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2\sqrt{2}-2 & -2\sqrt{2}-2 \end{bmatrix}^{-1}.$$

Inserting this into our expression for $\begin{bmatrix} P_n \\ Q_n \end{bmatrix}$, and using the matrix inverse

$$\begin{bmatrix} 1 & 1 \\ 2\sqrt{2}-2 & -2\sqrt{2}-2 \end{bmatrix}^{-1} = \frac{1}{-4\sqrt{2}} \begin{bmatrix} -2\sqrt{2}-2 & -1 \\ -2\sqrt{2}+2 & 1 \end{bmatrix},$$

we end up with

$$\begin{aligned} P_n &= \frac{1}{4\sqrt{2}\cdot 6^n}[(2+2\sqrt{2})^{n-1}(4+2\sqrt{2}) + (2-2\sqrt{2})^{n-1}(-4+2\sqrt{2})] \\ &= \frac{1}{4}\left(\frac{1+\sqrt{2}}{3}\right)^n + \frac{1}{4}\left(\frac{1-\sqrt{2}}{3}\right)^n. \end{aligned}$$

8. Does Dimension Really Matter?

Let G_n denote the set of all invertible $n \times n$ matrices with real entries. Is there a bijection (one-to-one and onto function) $f : G_2 \to G_3$ such that $f(AB) = f(A)f(B)$ for all A and B in G_2?

Solution 1. Suppose there were such a bijection f. Note that

$$f(I_2) = f(I_2 I_2) = f(I_2)f(I_2), \text{ so } f(I_2) = I_3,$$

where I_n denotes the $n \times n$ identity matrix. Now consider the three particular matrices

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \; Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \; R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{in } G_3.$$

Note that $PQ = R, P^2 = Q^2 = R^2 = I_3$. Therefore, if we put $A = f^{-1}(P), B = f^{-1}(Q), C = f^{-1}(R)$, we must also have $AB = C$, because $f(AB) = f(A)f(B) = PQ = R = f(C)$ and f is one-to-one, and similarly $A^2 = B^2 = C^2 = I_2$. In addition, because none of P, Q, R commute with all the matrices in G_3, none of A, B, C can commute with all the matrices in G_2. In particular, none of A, B, C can be I_2 or $-I_2$.

Consider the matrix equation $X^2 = I_2$. If $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ is a solution, then

$$x_1^2 + x_2x_3 = x_4^2 + x_2x_3 = 1\,,\; x_2(x_1 + x_4) = x_3(x_1 + x_4) = 0\,.$$

Thus, either $x_1 + x_4 = 0$ or $x_2 = x_3 = 0$. In the latter case, the matrix is diagonal with diagonal entries ± 1. In the former case, the trace of X is zero, so the eigenvalues sum to zero, and because the only possible eigenvalues are ± 1, there must be one of each. We can conclude that if $X^2 = I_2$, then either $\det(X) = -1$ or $X = I_2$ or $X = -I_2$. But that means that our matrices A, B, C must all have determinant -1, which contradicts $AB = C$. Therefore, there can be no such bijection.

Solution 2. There is no such bijection. If I_n is the identity in G_n, then $f(A) = f(AI_2) = f(A)f(I_2)$, so that $f(I_2) = I_3$. The eight elements

$$\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

all have square I_3 and commute with each other. The same must be true (with the squares being I_2) of the eight 2×2 matrices

$$S = f^{-1}\left(\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}\right).$$

The only eigenvalues of elements of S are ± 1. If a matrix in S has 1 as a double eigenvalue, it is similar to

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}.$$

Its square is I_2 if and only if $x = 0$, which means the original matrix is I_2. Similarly, if -1 is a double eigenvalue of a matrix in S, it is $-I_2$. Choose $A \in S$ other than $\pm I_2$. It must be diagonalizable with eigenvalues ± 1. If

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

then the eight matrices in $P^{-1}SP$ commute and have square I_2. If

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

commutes with

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

then $x = y = 0$. If its square is I_2, then $w = \pm 1$, $z = \pm 1$. Thus, $P^{-1}SP$ contains at most four matrices, not eight as needed.

Note. The end of the second solution can be shortened by using the theorem that commuting, diagonalizable matrices are simultaneously diagonalizable.

9. A Fine Line Between Convergence and Divergence

Let $\left(a_n\right)_{n\geq 1}$ be a sequence of positive numbers that have limit 0. Must there exist a sequence (b_n) of positive numbers such that $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges and $\displaystyle\sum_{n=1}^{\infty} a_n b_n$ converges?

Solution 1. There must be such a sequence. Because $(a_n)_{n\geq 1}$ converges to 0, it must be bounded above by some constant c. Furthermore, each of the sets

$$S_k = \{n \big| \, c/2^{k+1} < a_n \leq c/2^k\}, \quad k = 0, 1, 2, \dots ,$$

is finite. Letting $|S_k|$ denote the cardinality of S_k, define $b_n = 1/|S_k|$, where $n \in S_k$. Then for S_k nonempty,

$$\sum_{n\in S_k} b_n = 1, \qquad \sum_{n\in S_k} a_n b_n \leq c/2^k.$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges, while $\displaystyle\sum_{n=1}^{\infty} a_n b_n \leq \sum_{k=0}^{\infty} c/2^k = 2c\,,$

showing that $\displaystyle\sum_{n=1}^{\infty} a_n b_n$ converges.

Solution 2. Here is another construction of such a sequence. Because $\lim\limits_{n\to\infty} a_n = 0$, we can find subscripts $n_1 < n_2 < n_3 < \dots$ such that

$$a_{n_1} < \frac{1}{2}\,, a_{n_2} < \frac{1}{4}\,, \dots, a_{n_k} < \frac{1}{2^k}\,, \dots .$$

Now define

$$b_n = 1 \text{ if } n = n_k \text{ for some } k\,,$$
$$b_n = \frac{1}{n^2} \text{ otherwise.}$$

Then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges because infinitely many of the terms are 1, while $\displaystyle\sum_{n=1}^{\infty} a_n b_n$ converges by comparison to

$$\sum_{k=1}^{\infty} \frac{1}{2^k} + \sum_{n=1}^{\infty} \frac{a_n}{n^2}\,,$$

where the second of the series added together converges by limit comparison to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

10. Pushing the e**nv**e**lop**e

Let d be a positive integer, and let S_d be the set of all the polynomials $p(x)$ of degree at most d such that all of $p(0), p'(0), p''(0), \ldots$ are nonnegative integers. Prove that there exists a polynomial $q_d(x)$ in S_d such that the following are equivalent for all $p(x)$ in S_d:

(i) $p(1/n) \le q_d(1/n)$ for all positive integers n;

(ii) $p(1/n)^n < e$ for all positive integers n.

Solution. Note that if there exists such a polynomial $q_d(x)$, it must be unique, and it will have the property that $q_d(1/n)^n < e$, and so $q_d(1/n) < e^{1/n}$, for all n. This suggests that $q_d(x)$ might be the degree d Taylor polynomial (near $x = 0$) for e^x, and in fact we will show that

$$q_d(x) = \sum_{k=0}^{d} \frac{x^k}{k!}$$

satisfies the conditions of the problem. First observe that the set S_d consists exactly of all polynomials of the form $\sum_{k=0}^{d} \frac{a_k}{k!} x^k$, where each a_k is a nonnegative integer; in particular, our polynomial $q_d(x)$ really is in S_d. It is easy to show (i) implies (ii): if $p(1/n) \le q_d(1/n)$, then

$$p(1/n)^n \le q_d(1/n)^n = \left(\sum_{k=0}^{d} 1/(n^k k!)\right)^n < \left(\sum_{k=0}^{\infty} 1/(n^k k!)\right)^n = \left(e^{1/n}\right)^n = e\,.$$

To show that (ii) implies (i), assume (ii), and suppose that $p(1/n) > q_d(1/n)$ for some positive integer n. Note that both $p(1/n)$ and $q_d(1/n)$ can be expressed as fractions with denominator $n^d d!$. Therefore, $p(1/n) \ge q_d(1/n) + 1/(n^d d!)$. On the other hand,

$$\begin{aligned}\sum_{k=d+1}^{\infty} 1/(n^k k!) < \sum_{k=d+1}^{\infty} 1/(n^k (d+1)!(d+2)^{k-d-1}) &= \frac{1}{n^{d+1}(d+1)!} \cdot \frac{1}{1 - \frac{1}{n(d+2)}} \\ &= \frac{1}{n^d d!} \cdot \frac{1}{n(d+1) - \frac{d+1}{d+2}} < \frac{1}{n^d d!} \cdot \frac{1}{d} \le \frac{1}{n^d d!}.\end{aligned}$$

Consequently, $p(1/n) > q_d(1/n) + \sum_{k=d+1}^{\infty} 1/(n^k k!) = e^{1/n}$ and $p(1/n)^n > e$, a contradiction; we are done.