The Joe Konhauser Problemfest

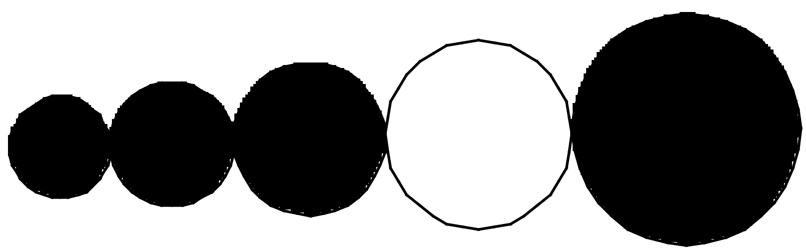
Macalester College April 17, 1993

- Proof or justification is required; it is not sufficient to simply state an answer.
- Calculators are allowed.

> This competition is dedicated to the memory of problem expert Joseph Konhauser, who died in February, 1992. Joe posted 682 Problems of the Week during 25 years at Macalester. Problems 1, 3, 5, and 10 are taken from the Konhauser collection.

1. Five Circles

Five circles are tangent to two nonparallel lines and to each other as illustrated. The smallest radius is 4; the largest radius is 9. What is the radius of the middle circle?



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2. Find the Rectangles

A regular 400-gon is tiled with nonoverlapping parallelograms. Prove that at least 100 of these parallelograms are rectangles.

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3. Pythagoras Meets Fermat

Show that Fermat's last theorem is valid for Pythagorean triples. In other words, prove that if p, q, and r are positive integers such that $p^2 + q^2 = r^2$, then $p^n + q^n \neq r^n$ for any integer n greater than 2.

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4. Random Queens

If two queens are randomly placed on distinct squares of an ordinary chessboard, what is the probability that they attack each other?

> NOTES. A chessboard has 8 rows and 8 columns. A queen attacks in all directions; she can move any number of squares in any row, column, or diagonal that contains her square.

5. Don't Cut Corners — Fold Them

Suppose the first quadrant of the x-y plane is a giant sheet of paper. Imagine that the corner at the origin is folded over onto the sheet in such a way that the triangle formed by the crease and the edges of the paper has constant area. Determine the locus of the corner. It suffices to find a description of this locus using equations of some sort along with a rough sketch.

6. Vectors

Are there six nonzero vectors in the plane that satisfy (a) and (b)?

(a) No two of the vectors lie on a line; this means, for example, that the set cannot have the vectors **V** and –2**V**, since they lie on one line.

(b) The vector sum of any two of the vectors is perpendicular to the sum of the other four.

7. A Nonzero Determinant

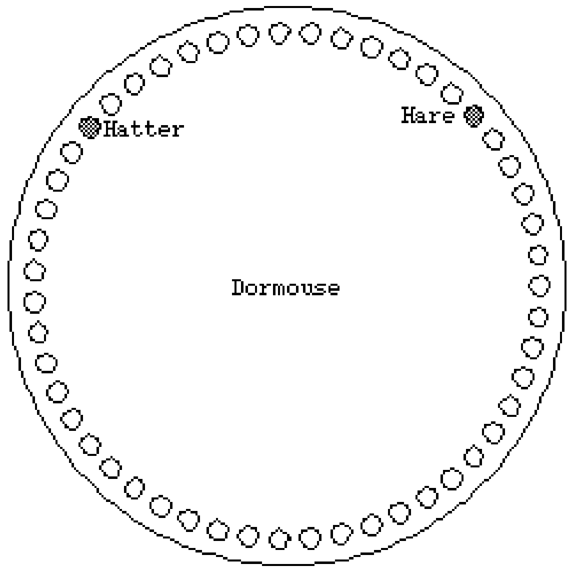
Suppose A is a 6×6 matrix with 0's on the main diagonal and with all other entries equal to either +1 or –1. Show that the determinant of A is nonzero.

8. How to Win at the Lottery

A $1 ticket in the Massachusetts Lottery consists of 6 different numbers chosen from 1, 2, 3, … , 48. On lottery day 6 numbers from this set are chosen at random (without repetition); a winning ticket is one that has at least 5 of these 6 numbers (order is irrelevant). Show that if one buys all tickets for which the sum of the entries is divisible by 47, then one is guaranteed of having a winner.

9. With Apologies to Lewis Carroll, Mathematician

While the Dormouse lay sleeping in the middle of the table, the Mad Hatter and the March Hare sat down for tea. The table had 51 places set and there were 13 places between the initial positions of our two characters, as illustrated. The Hatter and Hare immediately began moving, in turn, to other places according to the rules below.



1. A move is always to a place adjacent to the one being vacated.
2. The Hatter moved first and in a clockwise direction.
3. The Hare moved second, in a counterclockwise direction.
4. They always alternate moves, and each character moved when it was his turn.
5. If the Hatter and Hare arrive at adjacent seats, they both reverse direction on their next moves.
6. Directions never change except as specified in 5.

The party ended when the Hatter and Hare were again in their initial positions. After how many moves was the tea party over?

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10. An Elusive Quadratic

Find a second-degree polynomial with integer coefficients, $p(x) = ax^2 + bx + c$, such that p(1), p(2), p(3), and p(4) are perfect squares (that is, squares of integers), but p(5) is not.