SECOND ANNUAL KONHAUSERFEST

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(Problems by John Duncan, Univ. of Arkansas)

1. I stand vertically on a ski slope whose cross-section is known to be modeled by $y = -kx^n$ for some positive constant k and positive integer n with $n \geq 2$. I look downhill to the horizon point on the slope. To my surprise I find that, wherever I am on the slope, the distance to the horizon point -measured horizontally- is constant, say d. What is the value of n? Express my height in terms of k and d.

2. ABC is an equilateral triangle and P is a point on the minor arc of BC of the circumcircle. Show that PA = PB + PC.

3. Find det(A) for the n x n matrix $A = [|i - j|]$.

4. Let the sequence $\{x_n\}$ be defined by $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and for $n \geq 1$, x_{n+3} is the lat digit of $x_{n+2} + x_n$ (in base ten). Thus the sequence begins

   1, 2, 3, 4, 6, 9, 3, 9, ...

   In this sequence, will we ever find the successive triplet 2, 4, 6? Or the triplet 1, 1, 1?

5. Find the volume of the compact convex region given by

   $$K = \{(x,y,z): x^2 + y^2 \leq a^2, y^2 + z^2 \leq a^2, z^2 + x^2 \leq a^2\}$$

6. Twenty people sit equally spaced around a circular table. A 'party flip-flop' takes place as follows: choose a diameter of the circle and swap seats for each pair who are mirror images in the diameter. (Of course, no one moves for most choices of diameter.) Is it possible that after two party flip-flops each person has moved exactly one place forward around the table?

7. From a 7 x 7 chessboard I remove the middle square from oen edge. Is it possble to tile the remaining board with twelve T-tetrominoes? (A T-tetromino consists of four squares in the shape of a T, as shown.)

8. For a suitable choice of Cartesian basis, find the coordinates of the twenty vertices of a regular dodecahedron and show that any two non-parallel faces meet at an angle of arctan(2). (You are supplied with a model of a regular dodecahedron.)

9. A rook tours an 8x8 chessboard, passing through each square exactly once before returning to its initial square. Find the minimum number of rook moves needed to achieve this. What is the corresponding minimum number of rook moves for an 8x7 board.

10. Triangle PQR circumscribes the ellipse $x^2/2 + y^2 = 1$, Q lies on the line $x = -2$, and R lies on the line $x = 2$. Show that P lies on the ellipse $x^2/2 + y^2/9 = 1$.

Konhauser Problemfest 1994: SOLUTIONS

1. For general curve $y=f(x)$, the geometrical condition described says that

$$f(x) - [f(x-d) + h] = d\, f'(x).$$

For $f(x) = -kx^2$ this calls for

$$-kx^2 - [-k(x-d)^2 + h] = d[-2kx]$$

and this condition holds provided $h = kd^2$. For $n \geq 3$, it is obvious that the polynomial equation cannot hold for infinitely many values of x.

2. The obvious method is to use trigonometry (with or without complex number representations of P,A,B,C). Rather more elegantly, choose D on PA so that PD = PB. Angle properties in the circle and triangle give easily that PBD is equilateral. We readily deduce that triangles ADB and CPB are congruent. Hence AD = PC as required. (This result is the first step in a long and interesting story!)

3. There are several ways to see that $\det(A) = (-1)^{n-1}\, 2^{n-2}$. We can subtract successive rows, then do it again, and then expand. For an amusing, if perhaps not obvious variant, first move the first row down to the last row to get the matrix B - thereby introducing the factor $(-1)^{n-1}$. We can now factorize B as LU where L is unit lower triangular with subdiagonal exactly as in B, and U is upper triangular and copies B in the first two rows except for 0 in the (2,1)-position and thereafter is diagonal with 2 in each position except the final where the value is 2(n-1). The result follows.

4. Consider the given sequence mod 2; it becomes

1 0 1 0 0 1 1 1 0 1

which is clearly periodic and fails to contain 0 0 0. So the triplet 2 4 6 will never appear. There are only a finite number of distinct triplets and so eventually some triplet appears again (and then we cycle). But it is clear that every triplet has a unique preceding triplet, and so any initial triplet generates a cycle. Go backwards from 123 and we find the preceding portion as ....1 1 1 2 3. So 1 1 1 will appear.

[Problem 4 is a very special case of group action on $(Z_n)^k$, but it seems to be a non-trivial problem to specify the number or size of the orbits.]

5. Use the symmetry and wrestle with the visualization to see that

$$\text{Volume} = 16 \iint_S f(x,y)\, dx\, dy$$

where $f(x,y) = \min(\varnothing(x),\varnothing(y)$ with $\varnothing(t) = (a^2-t^2)^{1/2}$, and S is the sector of the disk $x^2 + y^2 \le a^2$ betweem y=0 and y=x. this gives

$$V/16 = \int_0^b dx \int_0^x \varnothing(x)\, dy + \int_b^a dx \int_0^{\varnothing(x)} \varnothing(x)\, dy \text{ (whre } b = a/\sqrt{2})$$

and eventually $V = 16 a^3 (1 - 1/\sqrt{2})$. We can also use polar coordinates, integrate first with respect to r and then knock the resulting trig integral into recognizable pieces. [This is a classical chestnut, not in as much favor as it should be. There are other elegant ways to do it, but the above are rather natural.]

6. When the obscurity of the model is seen through, we are just asking to write the cycle (1 2 3 ... 20) as a product of two involutions (an involution is a permutation equal to its own inverse). This is given by the involution [(19 1) (18 2) ... (11 9)] followed by the involution [(20 1) (19 2) ... (11 10)].

7. Label the 49 squares alternatively a,b. Without loss of generality, each corner square is labeled a and the middle of each edge is labeled b. Every T-tetromino covers either three a's and one b, or else one a and three b's. To get a nice *invariant* we want to choose a,b in a number system such that 3a+b=0 and a+3b=0. The determinant is 8 and so it is natural to work in Z_8. Take a=5 amd b=1. With one middle b removed we are left with 25 a's and 23 b's and this gives the total value of the board as 25a+23b = 125+23 = 148 = 4 (mod 8). But each tile has value 0 and so the tiling is impossible. [The corresponding problem with a corner square removd is of a different order of magnitude - the above invariant argument is inconclusive, and so also is the more sophisticated method of Conway & Lagarias.]

8. Place the doecahedron so that one edge only meets the table and the opposite edge is vertically above it. Regard the top edge as the ridge of a house with two ridge poles coming down from each end of the ridge. The four bottoms of the ridge poles form a horizontal square with a mirror square below. These eight vertices give a cube and may be coordinatized as (±1,±1,±1). Each regular pentagon face has diameter 2, and so by well known properties, each edge has length 2/y where y is the *Golden Mean* so that $y^2=y+1$. (If not well known, work out the angles in the pentagon/gram figure and use similar triangles.) One end of the top ridge thus is given by (0,1/y,c) and has distance $\sqrt{5} - 1$ from (1,1,1). After some algebra (a bit messy) we find that c=y. By symmetry we get the remaining twelve vertices in groups of four as (0,±1/y,±y), (±y,0,±1/y), (±1/y,±y,0).

   A dull cross=product calculation (or use centroids) now gets the normal vector for two adjacent faces, thence the cosine of the angle between them and thence acute angle as arctan 2. [Doubtless there is a more elegant method.]

9. Start at the top left, move to the top right, then bottom right, then one square left and now move up and down forming "curtains" until returning to the top left

square. For either size of board this requires 16 rook moves. We show first that this is minimal for the 8x7 case. We claim that in every column there is at least one vertical step. Otherwise each square is traversed horizontally - but for every right move there has to be a corresponding left move (to get back to the beginning!) and so one of the squares in the column must be traversed vertically. But every vertical move turns somewhere to a horizontal move and we annot go over the same square twice. So we must have at least eight horizontal moves, and hence at least 16 moves in total. For the 8x8 case, if there is one column that fails to have a vertical move (to stop the above proof), just turn the board through one right angle and there is a vertical move in every column!

[For the general $m^x n$ board, we need at least one of m,n even to be able to make such a tour, and then the minimum number of rook moves is just twice the least even number in {m,n}.]

10. By billiards ideas and Poncelet's theorem we know without computation that the locus is an ellipse an then easy calculation gets the ends of the major and minor axes. By affine geometry one could replace the given ellipse by the unit circle, but I still don't see how to use elementary circle geometry to show that if QR is tangent to the circle at (ε,η) then P is given by $(-\varepsilon,-k\eta)$. Here is one way to do the direct, messy calculation. Let Q = (-2,a), R = (2,b) and let QR be tangent to $x^2/2 + y^2 = 1$ at (ε,η). The tangent line $\varepsilon x/2 + \eta y = 1$ passes through (-2,a), (2,b) and this leads to $\eta = 2/(a+b)$, $\varepsilon = (a-b)/(a+b)$. Use the fact that (ε,η) lies on the ellipse to deduce that $a^2+6ab+b^2 = 8$. Let the other tangent from Q meet x=2 at (2,b*) and the other tangent from R meet x=-2 at (-2,a*). Then b* is the "other" root of $b^2+6ab+a^2 = 8$ and so b+b* = -6a, b* = -6a-b. In the same way, a+a* = -6b, a* = -a-6b. The other two tangent lines are:

$$(a-b^*0x+4y = 2(a+b^*) \text{ and } (a^*-b)x+4y = 2(a^*+b).$$

Substitute for a*,b* and we readily solve these equations to find that P is given by $(-\varepsilon,-\eta)$. The result is now evident.