Fourth Konhauser Problemfest

Macalester College Feb. 24, 1996

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Calculators of any sort are allowed. Justifications are expected, not just the statement of an answer (except in #2). Partial credit will be given for progress toward a solution or the answer to part of a question.

1. Leo's Erratic Bicycle

Leo can drive, bicycle, or walk from home to school in the morning.

- The drive takes 20 minutes.
- If he uses his bicycle, the trip takes 45 minutes.
- Once upon a time he walked, and it took 2.5 hours.

The evening before Leo has an important class at 8 AM, the weather forecast predicts that the next morning will be ideal for biking to school. Although Leo's bicycle is currently working perfectly, it has recently had frequent repair problems at random times.

Leo plans to bicycle the next morning. If the bicycle breaks down en route, from the point of the breakdown he will either:

- walk the rest of the distance to school, or
- walk back home and drive to school

(a). What is the latest minute (integer) when Leo must leave home the next morning to ensure he will be at school at or before 8 AM, even if his bike stops working some place en route?

(b). Suppose that he leaves at the time you specify in (a). He needs a plan of action in case the bike breaks down at a certain time. Give a time, to the nearest second, such that if the breakdown occurs at or before that time, his fastest way to school is to return home and drive, and if the breakdown occurs after that time, his fastest choice is to continue to school on foot.

2. An Unusual Set

The sum of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 is 45 and their product is 9!. Find a different choice of nine integers from {1, 2, 3, 4, 5, 6, 7, 8, 9}, necessarily with repetitions, whose sum is 45 and whose product is 9!.

> NOTE. Details of your method are not required, but we would be interested in seeing any rough work that was relevant to the search.

3. Dodecagon Diagonals

Let ABCDEFGHIJKL be a regular 12-sided polygon. TRUE or FALSE: The diagonals AE, CF, and DH intersect in a point.

4. Fractions from Sequences

Suppose $\{a_n\}$ is a nonconstant infinite arithmetic progression of real numbers, and let

$$b_n = \frac{a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n}{a_{n+1} + a_{n+2} + a_{n+3} + \cdot \cdot \cdot + a_{2n}} .$$

Assume that the a_n are such that the denominator of b_n is never 0.

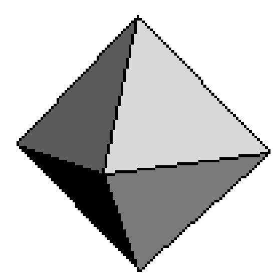
(a). Which real numbers can arise as $\lim_{n\to\infty} b_n$, as $\{a_n\}$ varies over all possibilities?

(b). Describe (with proof) the set of those $\{a_n\}$ as above for which $\{b_n\}$ is a constant sequence.

NOTE. An *arithmetic progression* is a sequence for which there is a constant such that each term after the first is obtained by adding the constant to the preceding term.

5. Random Octahedral Walk

A bug takes a random walk along the edges of a regular octahedron. At each vertex, the bug randomly chooses one of the four edges that come together at that vertex, each edge being equally likely, and walks along that edge to the vertex at the other end. Let $f(n)$ be the probability that, after traversing n edges, the bug is at the vertex at which it started.



Find rational numbers a, b, and c such that, for $n > 1$, $f(n) = a + b\,c^n$.

NOTE. A regular octahedron is a solid having eight faces that are equilateral triangles, with four coming together at each vertex.

6. Greatest Greatest Common Divisor

What is the largest possible value of the greatest common divisor of $n^2 + 1$ and $(n + 1)^2 + 1$ as n ranges over the positive integers?

7. A Truly Diophantine Problem

(a). Find 3 distinct positive integers such that the sum of every two of them is a square.

(b). Find 4 distinct positive integers such that the sum of every three of them is a square.

8. Two, and Only Two

Find a point P inside the triangle (0, 0), (1, 0), and (0, 1) such that there are exactly two lines through P that divide the area of T into two equal portions.

9. How Low Can You Go?

Find a function $y = f(x)$ which is not identically zero on the interval [0, 1], but for which $f(0) = 0 = f(1)$

and with $\dfrac{\int_0^1 f'(x)^2\,dx}{\int_0^1 f(x)^2\,dx}$ as small as you can make it.

> NOTE. This problem will be scored on a curve; the smaller the value for the ratio of integrals, the better the score. Thus you should submit something, for it might be worth full credit. However, note that a proof of minimality is *not* asked for.

10. Wronskians of Wronskians

For any sequence of functions $\{f_1, f_2, \dots, f_n\}$, all of which are sufficiently differentiable, let

$$W(f_1, f_2, \dots, f_n) = \det \begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ f_1' & f_2' & f_3' & \cdots & f_n' \\ f_1'' & f_2'' & f_3'' & \cdots & f_n'' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix}$$

where $f^{(i)}$ is the ith derivativeof f. This is called the *Wronskian determinant* of the sequence.

If $\{g_1, g_2, g_3, g_4, g_5, g_6\}$ is a sequence of positive functions, all of which are sufficiently differentiable, then there are integers a_i, $i = 1, 2, 3, 4, 5, 6$, such that

$$W(W(g_1, g_2), W(g_1, g_3), W(g_1, g_4), \dots, W(g_1, g_6)) = g_1^{a_1} g_2^{a_2} g_3^{a_3} \cdots g_6^{a_6}\; W(g_1, g_2, \dots, g_6)$$

Find, with proof, the integers $a_1, a_2, a_3, a_4, a_5, a_6$.

> NOTE. Partial credit will be given if the corresponding problem is written and solved for some integer $k \geq 3$. The bigger the k, the better, of course!