Eighth Annual Konhauser Problemfest

Macalester College **Feb. 26, 2000**

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Calculators of any sort are allowed. Justifications are expected, not just the statement of an answer. Partial credit will be given for progress toward a solution or the answer to part of a question.

1. Average Age The average age of the students in Professor K's class is 19.5 years and the average age of the men in the class is 19.7 years. Is it possible for the average age of the women to be also greater than 19.5?

2. A Normal Problem Determine the values of the parameter a for which there exists a line that is tangent to the graph of the curve $y = x^3 - a x$ at one point and normal to the graph at another. ("Normal" means perpendicular to the tangent.)

3. Win at Solitaire Consider the following game of solitaire: 26 red cards and 26 black cards are shuffled and separated into two equal piles labelled R and B; note that each of the piles R and B may contain cards of either color. The top card is drawn from pile R. The game proceeds by repeating the following process:

> If the top card taken from a pile is red, then the card is discarded and the next card is then taken from the top of pile R; if the top card taken from a pile is black, then the card is discarded and the next card is taken from the top of pile B.

The game ends when the last red card is taken, at which point the pile R is exhausted. What is the probability that all the cards in B have been picked up and discarded at this stage?

4. One Triangle Equals Two *ABCD* is a rectangle and *APQ* is an inscribed equilateral triangle for which P lies on *BC* and Q lies on *CD*.

(a) For which rectangles is this configuration possible?

(b) Prove that, when this configuration is possible, then the area of D*CPQ* is equal to the sum of the areas of triangles *ABP* and *ADQ*.

5. Cross This Bridge Four people A, B, C, D walking late at night with a flashlight come to a small rickety bridge. Only two can cross at a time, walking together with the flashlight, and one must return with the flashlight until all have crossed. They walk at different maximum speeds, requiring, respectively, a, b, c, d minutes to cross the bridge, where $a > b > c > d > 0$. When two walk together, each walks at the pace of the slower.

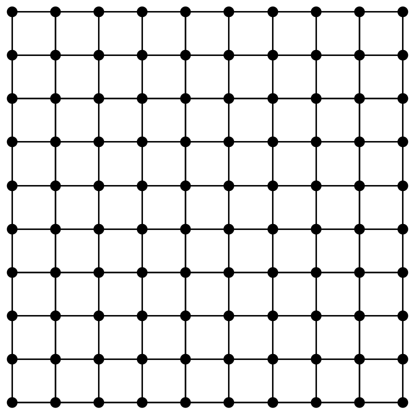
Determine necessary and sufficient conditions on a, b, c, d so that the minimum time to get all four across the bridge is $a + b + c + 2d$.

6. Color the Whole Lattice Consider the infinite integer lattice in the plane as a graph (see figure), with edges being the lines of unit length containing nearby points. What is the minimum number of colors that can be used to color all the vertices and edges of this graph, so that

(a) each pair of adjacent vertices gets two disinct colors;

(b) each pair of edges that meet at a vertex gets two distinct colors;

(c) an edge is colored differently than either of the two vertices at the ends?



7. Repeating Digits **(a)** Determine all positive integers n for which there exists an n-digit number divisible by 2^n with all digits the same.
(b) Determine all pairs (a, b) of nonzero digits such that, for each positive integer n, there exists an n-digit number divisible by 2^n whose digits are a and b.

8. Linear Independence Let $\boldsymbol{V}$ be the vector space of all continuous real-valued functions defined on the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, with the sum of two functions and the product of a function and a real scalar defined in the usual way.

(a) Prove that the four functions $\sin x$, $\cos x$, $\tan x$, and $\sec x$ are linearly independent.

(b) Let $\boldsymbol{W}$ denote the linear span of the four functions $\sin x$, $\cos x$, $\tan x$, and $\sec x$. Let T be the linear transformation to $\mathbb{R}^1$ defined on all of $\boldsymbol{W}$ by defining $T(\sin x) = \sin^2 x$, $T(\cos x) = \cos^2 x$, $T(\tan x) = \tan^2 x$, and $T(\sec x) = \sec^2 x$. Find a basis for the kernel of T. (The *kernel* is the set of vectors in $\boldsymbol{W}$ that T maps to 0.)

9. Find the Maximum Find the maximum value of $(1 - x)(1 - y)(1 - z)$ for nonnegative values of x, y, and z on the unit sphere $x^2 + y^2 + z^2 = 1$.

10. Sqrt, Floor, and Frac Suppose that $x \geq 1$, and that $x = \lfloor x \rfloor + \{x\}$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x and the fractional part $\{x\}$ satisfies $0 \le \{x\} < 1$. Define

$$f(x) = \frac{\sqrt{\lfloor x \rfloor} + \sqrt{\{x\}}}{\sqrt{x}}.$$

(a) Determine the supremum of the values of $f(x)$ for $1 \le x$.

(b) Let $x_0 \geq 1$ be given, and for $n \geq 1$ define $x_n = f(x_{n-1})$. Prove that $\lim_{n \to \infty} x_n$ exists.