Tenth Annual Konhauser Problemfest

University of St. Thomas February 23, 2002.

Calculators of any sort are allowed. Justifications are expected, not just the statement of an answer. Partial credit will be given for substantial progress towards a solution.

1. Cosine Values.

a) Prove that $\cos\frac{\pi}{9}$ is a solution of the equation $8x^3 - 6x = 1$.

b) Prove that $\cos\left(\frac{2\pi}{5}\right) = \frac{-1+\sqrt{5}}{4}$.

2. Function Equation.

Find all functions $f : \boldsymbol{R} \to \boldsymbol{R}$ satisfying $3f(x-1) + f(1-x) = 4x$ for all $x \in \boldsymbol{R}$.

3. An Upper Bound.

Let $x_1, x_2, \ldots, x_n$ be arbitrary numbers selected from the interval [0, 1] with $n \geq 2$. Show that

$+ x_2 + \ldots + x_n - x_1x_2 - x_2x_3 - \ldots - x_{n-1}x_n - x_nx_1 \leq \left\lfloor \frac{n}{2} \right\rfloor$ where $\left\lfloor \frac{n}{2} \right\rfloor$ represents the greatest integer not exceeding $\frac{n}{2}$.

4. Multiples of 13?

a) Show that $3^{2010} + 5^{2010}$ is divisible by 13.

b) Show that $3^{2004} + 5^{2004}$ is not divisible by 13.

5. An Integral Inequality.

Let f be a continuous function so that $f(x+1) = f(x)$ for all $x \in \boldsymbol{R}$. Show that there exists a number $s \in \boldsymbol{R}$ so that for all $x \in \boldsymbol{R}$, $\int_0^x f(s+t)\,dt \leq x\int_0^1 f(t)\,dt$.

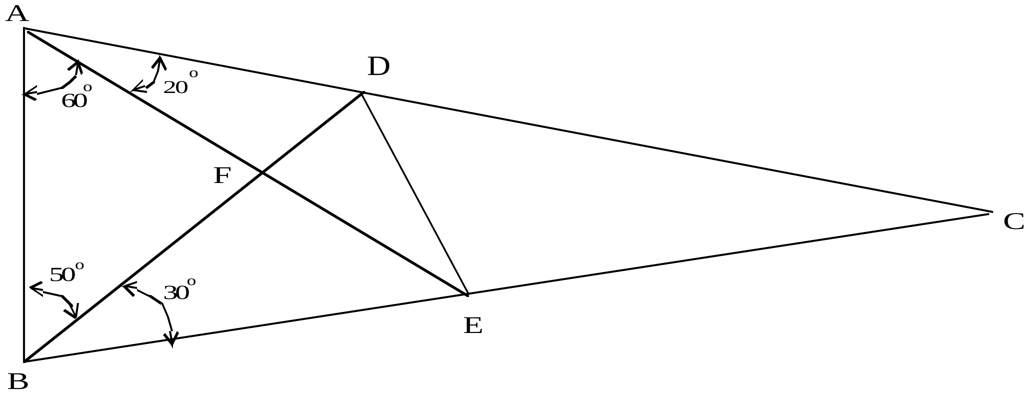
6. Serrin's Integral Formula.

Let f be a continuously differentiable function on the interval [0, 1] with $f(0) = 0$. Show that

$$\int_0^x\int_0^v\int_0^u \frac{2f(t)f'(t)}{1-t^2}\,dt\,du\,dv = \int_0^x \frac{(x-t)(1-xt)f(t)^2}{(1-t^2)^2}\,dt$$

7. Find the Angle.

Consider the isosceles triangle ABC below and the given angles (not necessarily drawn to absolutely correct scale). Find (with proof) the measure of angle DEF .



8. Sequence of Flips.

Consider the experiment of flipping a fair coin an infinite number of times with the various flips being independent. Here are two examples from among the infinite number of outcomes of the experiment:

H H T T H T T H T H ... ,   T T H T H T T H T H H H H T H T H T T .....

where ‘H’ denotes ‘heads’ and ‘T’ denotes ‘tails’. For the first of these outcomes, we notice that H T T (on flips 2, 3, and 4) occurred before T T H (on flips 3, 4, and 5), which in turn occurred before H T H (on flips 8, 9, and 10). For the second outcome, T T H (on flips 1, 2, and 3) occurred before H T T (on flips 5, 6, and 7).

In this experiment,

a) What is the probability that T T H occurs before H T T?

b) What is the probability that H T T occurs before H T H?

c) What is the probability that T T H occurs before H T H?

9. Guess The Number.

Ten students tried to guess the 5-digit number that their teacher was thinking of. Nobody succeeded, but each person did get one, and only one, of the 5 digits correct in the right position. What is the teacher's 5-digit number? Here are the 10 guesses:

06432, 29751, 94700, 38977, 87036, 43069, 76330, 52025, 61825, 18641

10. Upper Bound.

Let $\{a_1, a_2, \ldots\}$ and $\{b_1, b_2, \ldots\}$ be sequences of positive real numbers such that $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = 0$. Show that for large values of n,

$$a_n b_n < \max\left\{\frac{a_n}{\ln\frac{1}{a_n}},\, e^{-\frac{1}{b_n}}\right\}$$