The Thirteenth Annual Konhauser Problemfest

Carleton College, February 26, 2005

Problems set by Loren C. Larson

This contest is held annually in memory of Professor Joseph Konhauser (1924-1992) of Macalester College, who posted nearly 700 Problems of the Week at Macalester over a 25-year period. Joe died in February of 1992, and the contest was started the following year.

INSTRUCTIONS: Each team must hand in all work to be graded at the same time (at the end of the three-hour period). Each problem must be written on a separate page (or pages) and YOUR TEAM NAME SHOULD APPEAR AT THE TOP OF EVERY PAGE. Only one version of each problem will be accepted per team. Calculators of any sort are allowed. Justifications and/or explanations are expected for all problems. All ten problems will be weighted equally, and partial credit will be given for substantial progress toward a solution.

1. **Computing Today**

   Jan has a computer that admits just two operations: Given a number x it can compute $2x$ (label this operation D), and $2x-1$ (label this operation $\overline{D}$). Starting from 1, show how Jan can compute 2262005. Your answer should be a string of Ds and $\overline{D}$s where, starting from 1, the operations are performed left to right.

2. **Largest vertical strip**

   For what nonnegative number x does $F(x) = \int_x^{x+1} \sqrt{t}e^{-t}dt$ take on its maximum value? Give reasons for your answer.

3. Cross-ratio

The diagonals AC and BD of a quadrilateral $ABCD$ intersect at E inside the quadrilateral (i.e., the quadrilateral is convex). The quadrilateral is divided into four triangles ABE, BCE, CDE, and DAE. Let these triangles have areas S_1, S_2, S_3, S_4, not necessarily in that order, with $S_1 \geq S_2 \geq S_3 \geq S_4$. Find the possible values of $\dfrac{S_1 S_4}{S_2 S_3}$.

4. Definite integral

Let $G(x)$ be the solution of the differential equation $\dfrac{dy}{dx} = \dfrac{1}{5y^4 + 1}$, with $G(1) = 0$. Find $\displaystyle\int_1^3 G(x)\,dx$.

5. Well-balanced sets

The sets $\{1, 8, 12\}$ and $\{2, 3, 16\}$ have the same sum $(1 + 8 + 12 = 2 + 3 + 16 = 21)$ and the same product $(1 \cdot 8 \cdot 12 = 2 \cdot 3 \cdot 16 = 96)$. Prove that for any positive integer $n \geq 3$, there are *disjoint* sets, each with n *different* positive integers, which have the same sum and the same product.

6. L.A. Area

Let L_a denote the set of points on the line segment joining the points $A_a = (a, a^2)$ and $A_{a+1} = (a + 1, (a + 1)^2)$. Find the area of the region $\displaystyle\bigcup_{-1 \leq a \leq 0} L_a$. (By definition, this union consists of the points (x, y) that are in at least one of the sets L_a with $-1 \leq a \leq 0$.)

7. Divergent series

Define a sequence of numbers by setting $a_1 = a_2 = 1$, and for $n \geq 2$, $a_{n+1} > 0$ and

$$a_{n+1}^2 = 1 + 2\left(\frac{a_2}{a_1} + \frac{a_3}{a_2} + \cdots + \frac{a_n}{a_{n-1}}\right) + \frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_{n-1}^2}.$$

In particular, $a_3 = 2$. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges.

8. Navigable superknights

Let m and n be positive integers. From the point (x, y) in the plane, one is allowed to move to any of the points $(x \pm m, y \pm n)$ and $(x \pm n, y \pm m)$, where the plus/minus signs are independent of each other. For what values of m and n can one start at the origin and reach any point in the plane with integral coordinates by a succession of moves of the above type? Justify your answer.

9. Three-step shuffle

Let $a_k = 2$ if k is a multiple of 3 and $a_k = -1$ otherwise. Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^k a_k}{k} = 1 - \frac{1}{2} - \frac{2}{3} - \frac{1}{4} + \frac{1}{5} + \frac{2}{6} + \cdots$$

(You may assume that the series converges.) Justify your answer.

10. Sums of digits

Let $s(n)$ denote the sum of the digits of n, where the positive integer n is written in base 10. Is there an n such that $s(n) = 2005$ and $s(n^2) = 2005^2$? Show that your answer is correct.