

The Fourteenth Annual Konhauser Problemfest

University of St. Thomas Feb. 25, 2006.
Problems set by Bert Fristedt (Univ. of Minnesota)

1. Triangle Points

It is a theorem of plane Euclidean geometry that the three altitudes of any triangle meet at a point; this point is called the orthocenter of the triangle. [An altitude of a triangle is a line through a vertex perpendicular to the opposite side (possibly extended beyond the triangle).]

(A) Verify the theorem for a triangle with vertices at $(0, 0)$, $(a, 0)$ for an arbitrary $a > 0$, and (c, d) , for arbitrary $d > 0$ and c ; that is, show that the three altitudes meet. Also give the coordinates of the orthocenter.

(B) The centroid of this triangle is at $((a + c)/3, d/3)$. Prove that the orthocenter and the centroid are the same point if and only if the triangle is equilateral.

2. Pythagorean Squares

It is well known that there are infinitely many solutions of the equation $a^2 + b^2 = c^2$ in positive integers a , b , and c having greatest common divisor equal to 1. Moreover, it is known that for any such solution, c is an odd integer; and that it follows that c^2 is an odd integer. Prove, possibly using the facts just mentioned, that the equation

$$a^2 + b^2 + c^2 = d^2$$

Has infinitely many solutions in positive integers a , b , c , and d having greatest common divisor equal to 1 and which satisfy the additional condition that c and d are consecutive integers.

3. Orthogonal Tangents

Find all points (u, v) such that there are two tangent lines to the parabola $y = x^2$ that meet at a right angle at (u, v) . Show your calculations and write your answer in a simple form.

4. Infinite Series

Evaluate $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+2)(n+4)}$ as a specific real number, ∞ , or $-\infty$ — or else conclude that this infinite series does not have a value, even in this extended sense.

5. Limit of a Function

Given: The domain of a certain function f is the entire real line, $f(0) = 1$, and

$$f'(x) = \frac{|x^2 - 1|}{x^2 + 1}$$

For all x . Calculate $\lim_{x \rightarrow \infty} (f(x) - x)$

6. Recursive Sequences

(A) Let the sequence $\{a_n, n = 0, 1, \dots\}$ be defined recursively via

$$a_0 = 0 \quad \text{and} \quad a_{n+1} = -\frac{6}{a_n^2 - 7} \quad \text{for } n \geq 0.$$

Decide if $\lim_{n \rightarrow \infty} a_n$ exists as a real number. If so, evaluate it. If not, decide which of the following is true: it equal $+\infty$, it equals $-\infty$; or neither of these. [You may take is as given that in applying the recursion, the situation of a denominator equaling 0 never arises.]

(B) Do the same as for (A) but do it for the following sequence obtained by changing the initial conditions from that in (A):

$$b_0 = 8/3 \quad \text{and} \quad b_{n+1} = -\frac{6}{b_n^2 - 7} \quad \text{for } n \geq 0.$$

7. Pyramid Angles

For the square pyramid whose four lateral faces are equilateral triangles, calculate the measure of the angle at which each of the lateral faces meets the square base and also the measure of the angle at which any two adjacent lateral faces meet. You may leave an inverse trigonometric function in each answer.

8. Nonnegative Integers

For a fixed but arbitrary positive integer n , set $N = n(n + 1)/2$ and let \mathcal{S} denote the set of all nonincreasing infinite sequences $X = (x_1, x_2, \dots)$ of nonnegative integers satisfying

$$\sum_{i=1}^{\infty} x_i = N.$$

For $i = 1, 2, \dots$, set

$$f_i(X) = \begin{cases} 1 + x_{i+1} & \text{if } i \leq x_1 \\ x_{i+1} & \text{if } i > x_1 \end{cases},$$

And then set

$$F(X) = (f_1(X), f_2(X), , \dots).$$

You may take the following two facts as known:

- $F(X) \in S$ for all $X \in S$
- $F(C) = C$ for the particular sequence

$$C = (n, (n-1), \dots, 2, 1, 0, 0, 0, \dots).$$

[that is, for $c_i = \max\{n+1-i, 0\}$].

You may do the three parts of this problem in any order.

(A) Prove that F is not one-to-one if $n > 1$.

(B) Prove that $\sum_{i=1}^{\infty} |f_i(X) - c_i| \leq \sum_{i=1}^{\infty} |x_i - c_i|$.

(C) For $n = 5$ give an example that shows that strict inequality is possible in part (B).

9. Random Lengths

Begin with a stick of length 1. Break off a piece — call it piece A — randomly with its length being distributed uniformly on the interval $(0, 1)$. From the remaining piece A^c , whose length is one minus the length of piece A , break off a piece — call it piece B — randomly with its length being distributed uniformly over the length of piece A^c . [A precise way of saying this is that the conditional distribution of the length of B given the length of A is uniform on the interval from 0 to the length of A^c , but familiarity with conditioning terminology is not necessary in order to do this problem.] Let C denote the remaining piece.

There are six different permutations of the letters A , B , and C ; when this experiment is done one of these six permutations will match the order of the lengths from largest to smallest. [Thus BAC denotes an outcome in which B is the longest piece and C is the shortest piece (and there is no ambiguity because the probability is 0 that any two pieces will have exactly the same length).] Calculate the probabilities for each of the six permutations of A , B , and C .

10. Polynomial Roots

Find all six solutions of the equation $2z^6 + 3z^5 + 20z^4 + 27z^3 + 20z^2 + 3z + 2 = 0$.

Hint: There are no multiple roots, no rational roots, and no complex roots having rational real and imaginary parts.