Konhauser Problemfest
St. Olaf College, February 24, 2007
problems set by by David Molnar (University of Puget Sound)

1. **Who Ate My Pie?** Oh well, never mind, I've got this donut here. The thickness of the donut is one inch (that is, the donut is the solid of revolution obtained by rotating a circle of diameter 1 around the y-axis), and the volume is... well, I just ate the donut, I didn't know you were going to ask me the volume. But I can weigh myself, and having weighed myself beforehand and knowing the density of a donut, I can definitively tell you the volume of the donut was $8\pi^2$. But it would have been easier if you'd asked me earlier. What? Now you want to know the diameter of the donut? No, you can figure that out yourself.

2. **56 Straight** Let S be a set of six non-negative integers such that any integer from 1 to 56 is the sum of some subset of elements of S.

   a. Prove that 1, 2, and 4 must be in S.

   b. Find, with proof, the number of different sets S satisfying this criterion. (A list will not suffice without an argument that it is complete.)

3. **Why Intercept?** It is midnight. The Interceptor sits in the middle of the ocean, awaiting the appearance of a legendary ghost ship. The ghost ship will appear at some time within the next 24 hours (with the uniform distribution), and at a random location within the ocean (also with the uniform distribution). The ghost ship immediately heads directly towards the shore at its closest point. [Note that when I say, "The Interceptor sits in the middle of the ocean," I mean this quite literally – this is a circular ocean; the Interceptor is at the center of the circle.] Also, the speed of the Interceptor is such that it can reach the perimeter of the circle in exactly 12 hours; the speed of the ghost ship is half that. What is the probability that the Interceptor will reach the ghost ship before the ghost ship either reaches shore or disappears again at midnight tomorrow?

4. **Nerd Search** We are concerned with three-digit numbers hidden in the grid below appearing horizontally, vertically, or diagonally.

   a. (3 points) Find the longest sequence of consecutive three-digit numbers appearing in the grid. Each number must appear in a line, going one of the eight possible directions. The numbers themselves must be consecutive, but need not appear adjacent to each other in the grid.

   b. (7 points) Prove that your sequence is the longest possible.

4	1	8	2	7	9	5
1	8	5	4	0	2	1
8	2	1	9	8	7	3
8	7	6	8	1	9	2
7	0	5	4	9	5	7
3	8	4	6	2	1	8
2	7	3	8	1	8	1

5. **Blue Point Group** On the interior of a 4×4 square, 80 points are blue. No two blue points share an $x-$ or $y-$coordinate. Prove that some 1×1 square must have in its interior exactly 5 blue points.

6. **Paging Dr. Scholl's** There are six ways in which a person can put on his or her shoes and socks – for example, left sock, right sock, left shoe, right shoe. In how may ways can a centipede put on its shoes and socks? Assume that it has 100 feet, and different names for all of them.

7. **A Determined Composer** Let $\mathcal{A}$ be the set of all functions which are finite compositions of $x \mapsto x+1$, $x \mapsto x-1$, and $x \mapsto -\frac{1}{x}$. Show that $f \in \mathcal{A}$ if and only if $f(x)$ can be written as $\frac{ax+b}{cx+d}$ with a, b, c, d all integers and $ad - bc = 1$.

8. **Dots and Troxes** Bob admits to Alice that he is bored with Dots and Boxes and would like to play something new. Alice, who really enjoys Dots and Boxes, senses an opportunity.

"OK, Bob, ... if that is your real name... I'll just draw some dots anywhere on the paper, without making any three in a line. That will give the game more flexibility, don't you think? And, since you find keeping score so taxing, let's just make this a game one in which the player who makes the last legal move wins. And since I know you dislike ownership of material things, we'll say that a legal move consists of just drawing a line segment connecting two of the dots, without intersecting any previously-drawn segment. Nobody claims the boxes. Does that work for you?"

So Alice draws a few dots, and they play, and Alice wins. Bob is a bit suspicious of Alice being so nice and letting him go first, so for the second game he insists that she go first. So Alice draws some dots, they play the game, and Alice wins. Bob exclaims "I'm on to your tricks, Alice! I noticed you drew a different number of dots! This time we play with seven dots and I go first!" So Alice draws seven dots, Bob goes first, and Alice wins. After 10 or 20 games, Bob suddenly feigns having left a casserole in the oven.

Determine what Alice's secret is; that is, how no matter how many dots Bob insists they play with, and who he insists goes first, if Alice draws the dots, she wins.

9. **Number 5 Diagon Alley** Prove that, in any convex pentagon all of whose sides are length 1, the product of the diagonals is no greater than τ^5, where τ is the golden ratio $\tau = \frac{1+\sqrt{5}}{2}$.

10. Repetitive Harmonies Fix $a_1, \ldots, a_6$, and consider the infinite series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ where $a_n = a_{n-6}\ \forall n > 6$. Prove that this series converges if and only if $a_1 + a_2 + \ldots + a_6 = 0$.

Solutions
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1. The donut is a solid of revolution of a circle of radius 1/2 about the y-axis. Without loss of calories, let the center of that circle be at $(r, 0)$. A slice of the donut at height z is an annulus with outer radius $r + \sqrt{\frac{1}{4} - z^2}$ and inner radius $r - \sqrt{\frac{1}{4} - z^2}$. So, the volume of the donut is given by the integral

$$\int_{-1/2}^{1/2} \pi \left[\left(r + \sqrt{\frac{1}{4} - z^2} \right)^2 - \left(r - \sqrt{\frac{1}{4} - z^2} \right)^2 \right] dz$$

$$= \pi \int_{-1/2}^{1/2} 4r \sqrt{\frac{1}{4} - z^2}\, dz$$

$$= 4\pi r \int_{-1/2}^{1/2} \sqrt{\frac{1}{4} - z^2}\, dz$$

where the last integral is the area of a semicircle with radius 1/2, or $\pi/8$. Thus $8\pi^2 = (4\pi r)(\pi/8)$, or $r = 16$. We are asked for the *diameter* of the donut, which is a whopping **33** inches. Therefore, Larry Bird ate the pie.

2. Let S be a set of six positive integers, and let Z be the set of all sums of subsets of S. Note that since S is a *set*, its members are distinct, so we can write $S = \{a, b, c, d, e, f\}$ with $a < b < c < d < e < f$.

a. Assume that Z contains all integers from 1 to 56. If $1 \notin S$, then $1 \notin Z$. Similarly, since 2 cannot be written as a sum of *distinct* positive integers, $2 \in Z \Rightarrow 2 \in S$. Assume that $c = 3$. Then for 7 to be in Z, d must be 7 or less. Given $d \leq 7$, 14 can only be in Z if $e \leq 14$. Given $e \leq 14$, 28 can only be in Z if $f \leq 28$. But not we have $a+b+c+d+e+f \leq 1+2+3+7+14+28 = 55$, so $56 \notin Z$, a contradiction. Thus $c \geq 4$, but if $c \geq 5$, then $4 \notin Z$. So $c = 4$.

b. Given $a = 1, b = 2, c = 4$, we need to count triples (d, e, f) so that (using analysis similar to that above)

$$\begin{aligned} d &\leq 8 \\ e &\leq a+b+c+d+1 = 8+d \\ f &\leq a+b+c+d+e+1 = 8+d+e \\ 56 &\leq a+b+c+d+e+f \end{aligned}$$

If d is 5 or 6, then we quickly see that the last inequality is violated. So we have two cases.

Case 1: $d = 7$. Consider in the $e-f$ plane the region bounded by the lines $e = 15$, $f = 15+e$ and $e+f = 42$. There are six lattice points in this region, satisfying the given inequalities, giving the triples $(7, 14, 28), (7, 14, 29), (7, 15, 27), (7, 15, 28), (7, 15, 29)$, and $(7, 15, 30)$.

Case 2: $d = 8$. Consider in the $e - f$ plane the region bounded by the lines $e = 16$, $f = 16 + e$ and $e + f = 41$. There are twenty lattice points in this region. If $e = 13$, then f can be 28 or 29; if $e = 14$, then f can be 27, 28, 29, or 30; if $e = 15$, then f can be 26, 27, 28, 29, 30, or 31; if $e = 16$, then f can be 25, 26, 27, 28, 29, 30, 31, or 32.

This gives a total of 6+16=**26** such sets S.

3. Let the radius of the ocean be 1 (one ocean), and let us measure time in days rather than hours. The sample space consists of all points $(x, y, t) \in \mathbb{R}^3$ with $x^2 + y^2 \leq 1$ and $0 \leq t \leq 1$. This describes a cylinder C. Let P be the set of all points in C for which the Interceptor succeeds. Call the distance from the ghost ship to the Interceptor at the time the ghost ship appears r. If $r > 1/2$, then no matter what time this is, the ghost ship will reach the shore before the Interceptor can intercept it. So P is contained within a cylinder of radius 1/2. But there is also the time horizon to consider. The speed of the Interceptor is 2, and the speed of the ghost ship is 1. So if the Interceptor can catch up with the ghost ship, it will do so at time $t + r$. We must have $t + r \leq 1$, which describes a cone whose base is the same as that of C. P is thus a cylinder with radius 1/2 and height 1/2, surmounted by a cone with radius at its base 1/2 and height 1/2. Since the point (x, y, t) is chosen with the uniform distribution, the probability we want is Vol(P)/Vol(C), which is 1/8+(1/3)(1/8)=**1/6**.

4. a. The longest sequence consists of the **10** numbers from 180 to 189.

b. Color the squares in the grid red, white, or blue according to whether the digits are equivalent to zero, one, or two mod 3. There is *almost* a nice pattern: if we call the upper left square $(1, 1)$ and the lower right $(7, 7)$, then for the most part the digit in (i, j) is equivalent to $1 - (i + j)$ mod 3. If this were true of all the digits however, all three-digit numbers would consist of either one square of each color, or, on some diagonals, three squares of the same color. In either case such a coloring would force all three-digit numbers to be multiples of three, which would make part a pretty easy. As it is, there are a few exceptions to the rule; any three-digit number in the grid that is *not* a multiple of three must contain at least one of these digits. The digits that are not equivalent to $1 - (i + j)$ mod 3 are the 4 in the upper left corner, the three 8s along the diagonal in the upper left, and the two 8s in the lower right corner. Thus, *any three-digit number appearing in the grid that is not a multiple of three must contain the digit 8.*

If a sequence of ten or more numbers appeared whose *first* digit was 8, then at least one of these numbers would have to end in 0. But there are no 0s at a distance two from any of the 8s in the grid, so there are no such numbers. And there is no sequence of consecutive numbers in which some numbers have an 8 only in the tens place, and some have an 8 only in the hundreds place. So the sequence of length ten given is indeed the longest possible.

Alternate solution (Eric Egge) a. The numbers 180, 181, 182, 183, 184, 185, 186, 187, 188, and 189 all appear in the grid, and this is the longest sequence of consecutive three-digits numbers in the grid. To see where these numbers appear, number the rows from top to bottom and the columns from left to right. Then the following table gives the starting position of each of the numbers 180 through 189.

number	starting row,column	direction
180	4,5	up
181	7,7	left
182	1,2	down
183	7,5	left
184	3,3	up and left
185	2,1	right
186	4,5	left
187	7,7	up
188	2,1	down
189	3,3	down and right

b. Arguing by contradiction, suppose there were a sequence of length eleven in the grid. Then that sequence would contain a number ending in 0 and a number ending in 6, and these two numbers would have difference at most six. The three-digit numbers ending in 0 in the grid are 180, 270, 120, 540, 690, 780, 450, 840, and 960. The three-digit numbers ending in 6 in the grid are 876, 126, 516, 096, 186, 246, 456, 306, 846, 756, and 996. Therefore our sequence contains 180 and 186, 120 and 126, 450 and 456, or 840 and 846. Neither 179 nor 190 appears in the grid, which rules out 180 and 186. On the other hand, none of 123, 453, and 843 appear in the grid. (This is easy to check, since there are just three 3's in the grid, all of which are on an edge.) This is a contradiction, so there can be no sequence of length eleven in the grid.

5. Note that we can chop the 4×4 square into sixteen 1×1 squares, but it is not necessarily the case that any one of these squares contains exactly five blue points. Consider a 1×1 square (including its left and top boundaries, but not its right and bottom) situated in the upper-left corner of the larger square. We slide this square around horizontally and vertically according to the roadmap shown below, so that at time t it covers the small square labeled t. This defines a function $\chi : [0, 16) \to I\!N$, where $\chi(t)$ is the number of blue points in the small square at time t. Note that $\chi(0) + \chi(1) + \ldots + \chi(15) = 80$, since the sixteen translates of the small square at integer times cover the large square completely without overlap. (There are other ways to finesse this besides adding half the boundary to the small square, but it must be given some attention.)

00	01	02	03
15	06	05	04
14	07	08	09
13	12	11	10

If any of the sixteen values $\chi(n)$ is 5, we are done. Otherwise, for some integer i, we must have $\chi(i) > 5$, and for some integer j, we must have $\chi(j) < 5$. If only χ were continuous! However, it is in some sense as close to continuous as an integer-valued function can be; that is – because of the restriction that blue points cannot share a coordinate – for any a, $|\lim_{t\to a^-}\chi(t) - \lim_{t\to a^+}\chi(t)| \le 1$. That is, χ cannot jump by more than 1 at a time. Thus, in the interval $i < t < j$ (or $j < t < i$), there must be a time at which $\chi(t) = 5$.

Now there is the slight matter of the artifact introduced by adding part of the boundary to the square; if any of the five blue points in the 1×1 square found as above is on the upper or left edge, this will not satisfy the conditions of the problem. In such a case, we can translate the small square up and/or to the left by some distance δ smaller than the smallest of the (finitely many) distances from blue points to the boundary of this square, and then erase the boundary; this will not alter the count of the blue points in the small square.

6. The reason there are six ways for a person to put on his or her shoes is that there are six combinations of the letters LLRR, each of which can be read as a code to indicate the order in which to put on shoes and socks. (Whether we are to put on a shoe or a sock does not have to be part of the code – we put on a shoe if and only if that foot already has a sock on it.) Analogously, a string of 200 letters, two each of 100 different characters, can be read as a code for the centipede to put on its shoes and socks. There are $\mathbf{200}!/(\mathbf{2^{100}})$ such codes.

7. The 'easy' way first: to show that any function in $\mathcal{A}$ can be written in the prescribed form, we use induction on the number of functions in the composition. Let $\mathcal{A}_n$ denote the set of compositions of exactly n of the functions $x \mapsto x+1$, $x \mapsto x-1$ and $x \mapsto -\frac{1}{x}$. Let $I(n)$ be the statement that if $f \in \mathcal{A}_n$, then there exist integers a, b, c, d with $ad - bc = 1$ such that $f(x) = \frac{ax+b}{cx+d}$. $I(1)$ merely states that the three functions $x \mapsto x+1$, $x \mapsto x-1$ and $x \mapsto -\frac{1}{x}$ themselves have this property, which is self-evident. Let $f \in \mathcal{A}_n$. Then we can write $f = f_1 \circ f'$, where f_1 is one of the three given functions, and $f' \in \mathcal{A}_{n-1}$. By the induction hypothesis, $f'(x) = \frac{ax+b}{cx+d}$, with $ad - bc = 1$. If $f_1(x) = x+1$, then $f(x) = \frac{(a+c)x+(b+d)}{cx+d}$. $(a+c)(d) - (b+d)(c) = ad - bc = 1$, so f has the desired form. Similarly, if $f_1(x) = x - 1$, then $f(x) = \frac{(a-c)x+(b-d)}{cx+d}$, which also satisfies the condition. Last, if $f_1(x) = -\frac{1}{x}$, then $f(x) = \frac{-cx-d}{ax+b}$, and $(-c)(b) - (-d)(a) = ad - bc = 1$ as well. So in any case, f has the desired form, establishing $I(n)$. [The determinant $ad - bc$ is an invariant under composition by these three functions.]

In the other direction, we must show that if $f(x)$ can be written as $f(x) = \frac{ax+b}{cx+d}$, with $ad - bc = 1$, then $f \in \mathcal{A}$. Since we can simultaneously change the sign of all four of a, b, c, d, we can assume WLOG that $c \ge 0$. We proceed by induction on c. Let $J(n)$ be the statement that any rational function $\frac{ax+b}{cx+d}$, with a, b, c, d integers such that $ad - bc = 1$, and $|c| \le n$ can be written as a finite composition of $x \mapsto x+1$, $x \mapsto x-1$, and $x \mapsto -\frac{1}{x}$. $J(0)$ concerns functions of the form $f(x) = \frac{ax+b}{d}$, but for such a function to satisfy $ad - bc = 1$, it must be that (again, WLOG) $a = d = 1$, so $f(x)$ is composed of $|b|$ copies of either

$x \mapsto x+1$ or $x \mapsto x-1$.

Let $n > 1$, and $f(x) = \frac{ax+b}{nx+d}$. By the division algorithm, there exist integers q and r, $0 \le r < n$, so that $a = nq + r$. For these q and r, $\frac{ax+b}{nx+d} = q + \frac{rx+(b-dq)}{nx+d}$. Thus $\frac{ax+b}{nx+d}$ is a composition of $\frac{rx+(b-dq)}{nx+d}$ and $|q|$ copies of either $x \mapsto x+1$ or $x \mapsto x-1$. Consider $\frac{rx+(b-dq)}{nx+d}$ composed with $x \mapsto -\frac{1}{x}$, that is, $\frac{-nx-d}{rx+(b-dq)}$. By construction, r is non-negative and less than n. So, by the induction hypothesis, $\frac{-nx-d}{rx+(b-dq)}$ can be written as a finite composition of the three given functions. Thus $\frac{ax+b}{nx+d}$ can as well, establishing $J(n)$.

8. Since the game proceeds until there is no further legal moves, the final configuration of dots and line segments will form a connected planar graph in which all of the enclosed regions are triangles. Let v, e, and f be the number of vertices, edges, and faces of this graph respectively, let t be the number of triangles formed, and let m be the number of moves made in the game.

Note that every move consists of drawing one edge, so $m = e$. And t is merely $f - 1$, as the graph has one unbounded face. Let h be the number of vertices of that unbounded face. Summing the number of edges of each face, including the unbounded one, we get $3t + h$. But since each edge is incident to two faces, we know that $2e = 3t + h$. Along with Euler's Formula for planar graphs, we now have

$$\begin{aligned} v - e + f &= 2 \\ t &= f - 1 \\ 2e &= 3t + h. \end{aligned}$$

Eliminating f, we can then write

$$\begin{aligned} -e + t &= 1 - v \\ 2e - 3t &= h. \end{aligned}$$

The proper linear combination of these yields $m = e = 3v - 3 - h$. So, while there is no strategy whatsoever once the game starts, the winner depends on both the number of dots drawn and the number of these which are on the convex hull of the configuration of dots. Thus, if Alice is to go first, she needs to ensure that m will be odd, so h must have the same parity as v. If Alice is to go second, she wants m even, so v and h must be of opposite parity.

9. Let $\theta_1, \ldots, \theta_5$ be the angles of the pentagon. Note that $0 < \theta_j < \pi$ for every angle. Using Law of Cosines the five diagonals have lengths $\sqrt{2 - 2\cos(\theta_1)}, \ldots, \sqrt{2 - 2\cos(\theta_5)}$. Therefore we want to maximize $2^{5/2}\sqrt{(1 - \cos(\theta_1))(1 - \cos(\theta_2)) \ldots (1 - \cos(\theta_5))}$ subject to the constraint $\theta_1 + \ldots + \theta_5 = 3\pi$.

Setting the constant and the square root aside for the time being, take the logarithm of the quantity to be maximized;

$$L = \ln(1 - \cos(\theta_1)) + \ldots + \ln(1 - \cos(\theta_5)).$$

We want to show that $\sqrt{32e^L} \leq \tau^5$. Note that the function $\ln(1-\cos(\theta))$ is concave down on $[0,\pi]$: If $g(\theta) = \ln(1-\cos(\theta))$, then $g'(\theta) = \dfrac{\sin(\theta)}{1-\cos(\theta)}$, and $g''(\theta) = \dfrac{-1}{1-\cos(\theta)}$ which is negative on $(0,\pi)$. We can thus apply Jensen's Inequality which says that $\dfrac{\sum_{i=1}^n g(x_i)}{n} \leq g\Big(\sum_{i=1}^n \dfrac{x_i}{n}\Big)$, with equality only when all the x_i are equal. In this case we know that the sum $\theta_1+\theta_2+\theta_3+\theta_4+\theta_5 = 3\pi$, so $L/5 \leq \ln(1-\cos(3\pi/5)) = \ln\left(1 - \dfrac{1-\sqrt{5}}{4}\right) = \ln\left(\dfrac{3+\sqrt{5}}{4}\right)$. Therefore $L \leq \ln\left(\dfrac{3+\sqrt{5}}{4}\right)^5 = \ln\left(\dfrac{\tau^2}{2}\right)^5$, and $\sqrt{32e^L} \leq \tau^5$; this upper bound is obtained when the pentagon is regular and the diagonals are all length τ.

Alternate solution (Steve Tanner) Using the method of LaGrange Multipliers we want the gradient of the product function to equal λ times the gradient of the constraint function. That vector equation is equivalent to the following system of five equations:

$$32\sin(\theta_1)(1-\cos(\theta_2))(1-\cos(\theta_3))(1-\cos(\theta_4))(1-\cos(\theta_5)) = \lambda$$
$$32\sin(\theta_2)(1-\cos(\theta_1))(1-\cos(\theta_3))(1-\cos(\theta_4))(1-\cos(\theta_5)) = \lambda$$
$$32\sin(\theta_3)(1-\cos(\theta_1))(1-\cos(\theta_2))(1-\cos(\theta_4))(1-\cos(\theta_5)) = \lambda$$
$$32\sin(\theta_4)(1-\cos(\theta_1))(1-\cos(\theta_2))(1-\cos(\theta_3))(1-\cos(\theta_5)) = \lambda$$
$$32\sin(\theta_5)(1-\cos(\theta_1))(1-\cos(\theta_2))(1-\cos(\theta_3))(1-\cos(\theta_4)) = \lambda$$

The requirement $0 < \theta_j < \pi$ means that $(1-\cos(\theta_j)) > 0$, so we may look at each of these five equations pairwise and cancel like terms to get

$$\sin(\theta_i)(1-\cos(\theta_j)) = \sin(\theta_j)(1-\cos(\theta_i)) \ \forall i,j = 1,2,3,4,5.$$

This means that $\dfrac{\sin(\theta_i)}{1-\cos(\theta_i)}$ has the same value for every $i = 1,2,3,4,5$.

But this means that every one of the five angles is equal, for if we let $f(\theta) = \dfrac{\sin(\theta)}{1-\cos(\theta)}$ then $f'(\theta) = \dfrac{-1}{1-\cos(\theta)}$ which is negative on $(0,\pi)$. Therefore $f(\theta)$ is one-to-one on $(0,\pi)$, so each of the θ_i must be the same (and therefore equal to $3\pi/5$).

This is the unique point found by the LaGrange Multiplier method. It must be a maximum, since if we allow angles to take on endpoint values we get a product of zero. Therefore the product of the diagonals is maximized when the pentagon is regular.

10. Let the sequence $(a_n)_{n=1}^\infty$ be periodic with period 6; that is, $a_n = a_{n-6} \forall n > 6$. We wish to show that the series $\sum_{n=0}^\infty \dfrac{a_n}{n}$ converges if and only if

$$a_1 + a_2 + \ldots + a_6 = 0 \qquad (*)$$

Proof. Let $s_n = \sum_{j=1}^{n} \frac{a_j}{j}$.

First we show that when (*) holds, $(s_{6n})_{n=1}^{\infty}$ converges. Let Q be the sum of the positive a_i. Then replacing denominators of positive terms by $6n+1$ and of negative terms by $6n+7$ and vice versa,

$$\frac{Q}{6n+7} - \frac{Q}{6n+1} < \frac{a_1}{6n+1} + \frac{a_2}{6n+2} + \ldots + \frac{a_6}{6n+6} < \frac{Q}{6n+1} - \frac{Q}{6n+7}.$$

for any n. So, $|s_{6n}| = \left|\sum_{j=0}^{n-1}\left(\frac{a_1}{6n+1} + \frac{a_2}{6n+2} + \ldots + \frac{a_6}{6n+6}\right)\right| < \sum_{j=0}^{n-1}\left(\frac{Q}{6n+1} - \frac{Q}{6n+7}\right) = Q - \frac{Q}{6n+1} < Q$.

This established that the subsequence (s_{6n}) is bounded, but we still need to show that it is (eventually) monotone. The summands $b_n := \dfrac{a_1}{6n+1} + \dfrac{a_2}{6n+2} + \ldots + \dfrac{a_6}{6n+6}$ can change sign with n. Define on the interval $[0, \infty)$, $f(x) = \dfrac{a_1}{6x+1} + \dfrac{a_2}{6x+2} + \ldots + \dfrac{a_6}{6x+6}$. Then $f'(x) = -6\left[\dfrac{a_1}{(6x+1)^2} + \dfrac{a_2}{(6x+2)^2} + \ldots + \dfrac{a_6}{(6x+6)^2}\right] = -6\dfrac{g(x)}{\Pi(6x+j)^2}$, where $g(x)$ is a polynomial. Since the denominator is never zero, $f'(x)$ is continuous and has finitely many zeros, so by Rolle's Theorem, b_n can change sign only finitely many times. After the last of these sign changes, (s_{6n}) will be monotone. Thus (s_{6n}) converges.

Now to show that $(s_k)_{k=1}^{\infty}$ converges, take m to be the maximum of $\{|a_1|, |a_1 + a_2|, \ldots, |a_1 + \ldots + a_6|\}$. For any $k \geq 0$, write $k = 6q + r$, with $0 \leq r < p$. Then $|s_k - s_{6q}| < m/6q$. s_k is then sandwiched between $s_{6q} \pm m/6q$, and so since $m/6q \to 0$ as $k \to \infty$, $(s_k)_{k=1}^{\infty}$ converges to $\lim_{n\to\infty} s_{6n}$ as well.

Conversely, assume (*) does not hold. Let $D = a_1 + \ldots + a_6$. Then

$$\frac{a_1}{6n+1} + \frac{a_2}{6n+2} + \ldots + \frac{a_6}{6n+6} > \frac{Q}{6n+7} - \frac{Q-D}{6n+1} = \frac{D}{6n+1} - \frac{6Q}{(6n+1)(6n+7)}.$$

For sufficiently large n, $\frac{6Q}{6n+7} < 1/2$, so $\frac{D}{6n+1} - \frac{6Q}{(6n+1)(6n+7)} > \frac{1}{12n+2}$. So by the p-test the subsequence of partial sums $(s_{6n})_{n=1}^{\infty}$ diverges, and thus the series as well.

Alternate solution (Stan Wagon) Assume first that $a_1 + a_2 + \cdots + a_6 = 0$. Look at each group of six of the form

$$\frac{a_1}{6n+1} + \frac{a_2}{6n+2} + \frac{a_3}{6n+3} + \frac{a_4}{6n+4} + \frac{a_5}{6n+5} + \frac{a_6}{6n+6}.$$

Add these up, to get $\frac{P}{Q}$ where P has degree at most 4 (the degree-5 terms cancel) and Q has degree 6. Take absolute values and sum this expression as $n \to \infty$. The limit comparison test can be used, since $(P/Q)/(1/n^2) = Pn^2/Q$ has a finite limit and $\sum 1/n^2$ converges. So the sequence S of every sixth partial sum converges. Note that this means that the sequence of partial sums converge. For example, the sequence of partial sums indexed by $6n-1$ differs in each term from the the sequence S by $a_6/(6n+6)$, which goes to 0. Similarly for the other four subsequences. Therefore the given sequence of partial sums converges.

The same argument works if $a_1 + a_2 + \cdots + a_6 = c \neq 0$. Assume $c > 0$ (else multiply everything by -1). For then the grouped partial sums lead to P/Q where P has degree 5 and Q has degree 6 and the limit comparison test with the harmonic series yields divergence.