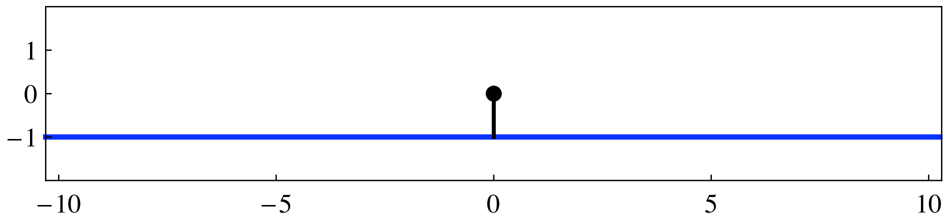
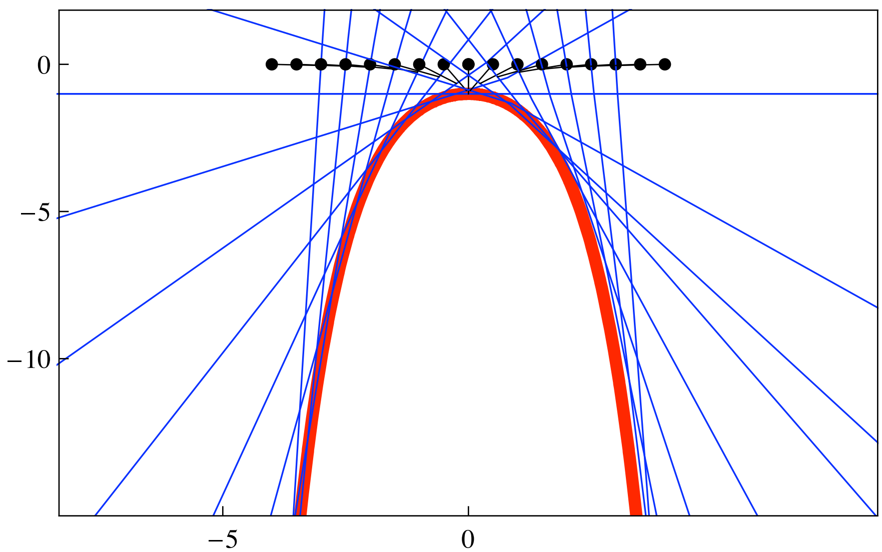
Why do catenaries work for the square wheel?

The main idea is that the inverted catenary works for a wheel that is simply an infinite straight line. Given that fact, one just cuts off the catenary where the tangent is at a 45° angle. This means two of them come together to form a right angle, which accepts the square corner perfectly. Here is the flat wheel, with its center considered to be at $(0, 0)$. In polar coordinates this straight line is $r = -\csc\theta$.



Here are several copies of the straight line as they roll along a catenary. Note that the dots representing the "center" of the straight-line "wheel" stay on a horizontal line.



Now, why does the catenary work for a straight line? Here are the highlights:

1. The polar form of the straight line wheel is $r = -\csc\theta$.

2. Let $\theta(x)$ represent the amount of angle the wheel has rolled when its center has horizontal coordinate x.

3. Let $y = f(x)$ denote the road shape, then $r(\theta(x)) = -f(x)$.

4. The fact that the arc lengths along the wheel and road must match leads to the differential equation $\frac{d\theta}{dx} = \frac{1}{r(\theta)}$, with initial conditions $\theta(0) = -\pi/2$. Separating the variables and using the cosecant form just given leads to $x = \int_{-\pi/2}^{\theta} r(\theta)\; d\theta$ or $x = \int_{-\pi/2}^{\theta} -\csc\theta\; d\theta$.

5. The integral can be evaluated as $x = -\log\left(-\tan\frac{\theta}{2}\right)$.

6. The result of the integration inverts to $\theta = 2\arctan\left(-e^{-x}\right)$

7. The road is then $y = f(x) = -r(\theta(x)) = \csc\left(2\arctan\left(-e^{-x}\right)\right)$, and this simplifies to $y = -\frac{e^{x}+e^{-x}}{2}$, an inverted catenary.