# Twelfth Annual Konhauser Problemfest

**Macalester College** **Feb. 28, 2004**

**All problems, except those credited otherwise, were created specifically for the Konhauser Problemfest or are variations of problems of Dirk Laurie used in other mathematics competitions.**

**1. Harmonic Arithmetic** The *arithmetic-harmonic* mean (AHM) of positive real numbers $x$ and $y$ is defined as follows:

> Let $a_0 = x$, $h_0 = y$, and inductively $a_{n+1}$ is the arithmetic mean, and $h_{n+1}$ the harmonic mean, of $a_n$ and $h_n$; that is: $a_{n+1} = \frac{1}{2}(a_n + h_n)$, $h_{n+1} = \left(\frac{1}{2}\left(a_n^{-1} + h_n^{-1}\right)\right)^{-1}$, $n = 1, 2, \ldots$. The two sequences $\{a_n\}$, $\{h_n\}$ have a common limit (you may take this fact as given), and AHM($x$, $y$) is this common limit.

What is AHM(4, 200)?

**Solution.** Examining the first few values of $a_n$ and $h_n$ will show that the product is unchanged through the iteration. This is easily proved:

$$a_{n+1}\,h_{n+1} = \left(\frac{a_n+h_n}{2}\right)\left(\frac{2}{a_n^{-1}+h_n^{-1}}\right) = \left(\frac{a_n+h_n}{2}\right)\left(\frac{2\,a_n\,h_n}{a_n+h_n}\right) = a_n\,h_n.$$

Therefore the limit $L$ satisfies $L \cdot L = a_0\,b_0$. For the given values $L = \sqrt{4 \cdot 200} = \sqrt{800} = 20\sqrt{2}$. Proving convergence was not required, but it can be done without difficulty.

**Comment.** The fact that the AHM is the same as the GM might account for its relative obscurity compared to the AGM, which is quite an important function, both historically and in modern numerical algorithms because of its super-fast convergence.

**2. Serious Implications** Complete the following logical crossword puzzle. Each square will contain either T (True) or F (False). The clues refer to the entire rows or columns, viewed as lists. So, for example, TFFF $\vee$ FFTF is TFTF.

| | e | f | g | h |
|---|---|---|---|---|
| a | | | | |
| b | | | | |
| c | | | | |
| d | | | | |

| ACROSS | | DOWN | |
|---|---|---|---|
| **a.** | $\neg$ f | **e.** | e $\Rightarrow$ b |
| **b.** | b $\vee$ b | **f.** | $\neg$ c |
| **c.** | b $\Rightarrow$ c | **g.** | $\neg$ (g $\Rightarrow$ ($\neg$ g)) |
| **d.** | a $\vee$ $\neg$h | **h.** | $\neg$ $\neg$ h |

**Solution.** This original problem is due to Jim Henle (Smith College); similar puzzles appear in his book *Sweet Reason* (Springer). The answer is:

| | e | f | g | h |
|---|---|---|---|---|
| a | T | F | T | T |
| b | T | T | T | T |
| c | T | F | T | T |
| d | T | F | T | T |

Because $A \Rightarrow B$ is logically equivalent to $B \vee \neg A$, the **e** clue can be written as **b** $\vee$ ($\neg$ **e**). Therefore no entry in the first column can be False, otherwise the disjunction would be True, contradiction. This means the first column is TTTT. And then the **e** clue tells us that the second row is TTTT. Now, because clues **b**, **g**, and **h** are tautologies, and therefore useless, and **e** is fully used, the clue set reduces to:

**a.** $\neg$ f  **c.** b $\Rightarrow$ c  **d.** a $\vee$ ($\neg$ h)  **f.** $\neg$ c

Then the **a** and **f** clues fill in everything except the positions $a_4$, $c_4$, $d_2$, $d_3$, and $d_4$. Clue **d** then makes $d_3$ true. And clue **d** implies that $d_4$, which equals $h_4$, is true. And clue **d** then makes $a_4$ true. Then $f_4$ is false and $c_4$ is true.

**3. Three Semicircles** Suppose $\Delta ABC$ has perimeter $p$ and three semicircles with diameters $AB$, $BC$, and $CA$ are drawn on the outside of $ABC$. Let $H$ be a circle that contains all three semicircles. Prove that the radius of $H$ is at least $p/4$.

**Solution.** Let $D, E, F$ bisect sides $BC$, $CA$, $AB$, respectively. Extend $DE$ in both directions so that it hits the semicircle on $BC$ in $P$ and the one on $AC$ in $Q$. Then $PQ = PD + DE + EQ = BD + AF + CE = (BC + AB + CA)/2 = p/2$. But $PQ$ lies entirely within the circle $H$, so of $r$ is the radius of the circle, $p/2 = PQ \le 2r$, and $r \ge p/4$.