

Nineteenth Annual Konhauser Problemfest

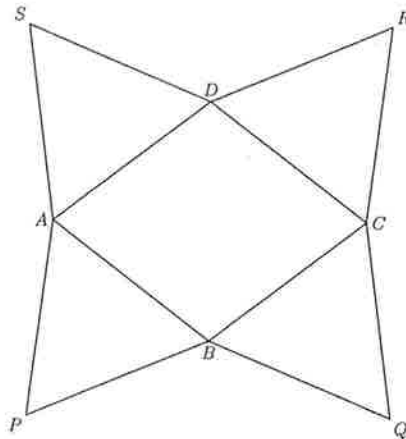
St. Olaf College

February 26, 2011

Problems by Andy Niedermaier (Jane Street Capital, New York City)

Calculators of any sort are allowed, although complete justifications are expected, not just the statement of an answer. Use of cell phones or computers is not permitted. Partial credit will be given for progress toward a solution or the answer to part of a question.

- 1. Smallinomial** The cubic equation $ax^3 + bx^2 + cx + d = 0$ has non-zero integer coefficients and distinct integer solutions. Find the smallest possible value for $|a| + |b| + |c| + |d|$.
- 2. The Traveling Ant** An ant walking on the plane departs from $(0, 0)$, traveling between lattice points. From any given lattice point (x, y) , the ant randomly decides to travel to $(x + 1, y)$, $(x, y + 1)$, or $(x + 1, y + 1)$. After some time, the ant arrives at $(4, 4)$. What is the probability that the ant stopped by $(2, 2)$ along the way?
- 3. Fickle Factorial** Find all integers n , $2 \leq n \leq 4010$, such that there exist integers x and y satisfying $2011! = \frac{x^2 y^3}{n!}$.
- 4. Determine It** Let A be a 2011×2011 matrix whose i, j entry is $(-1)^{i+j}$, and let I be the 2011×2011 identity matrix. Let $B(x) = A + Ix$, for $x \in \mathbb{R}$. Compute the determinant of $B(x)$.
- 5. Rhomboctagon** $ABCD$ is a rhombus with side length 13. Equilateral triangles are erected on all four sides, resulting in the concave octagon $APBQCRDS$. Given $PQ = 24$, compute the area of the octagon.



6. **Integral** Compute the exact value of $\int_0^{\pi/2} \ln \cos x \, dx$.

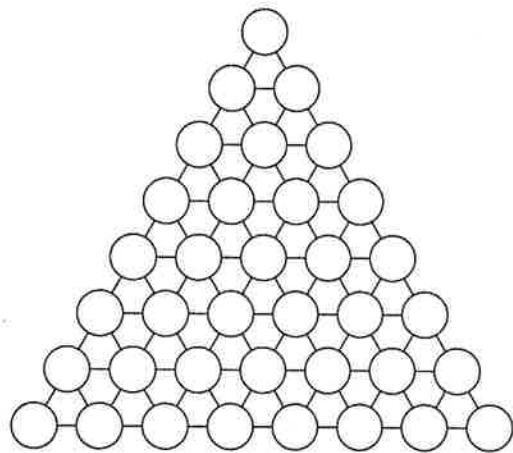
7. **Stay Positive** Show that there are no positive integer solutions to

$$a^4 + b^4 + c^4 + d^4 + e^4 = 16abcde.$$

8. **Stretch Yourself** An infinitely stretchable rubber band connects the rear of a truck to a post. The rubber band is 1 meter long at the start. A caterpillar starts walking along the rubber band, at a constant speed of 1 centimeter per second. At that same moment, the truck drives away from the post, at a constant speed of 100 meters per second. As the truck speeds away, the rubber band stretches uniformly. If the band never breaks, the truck never stops (or refuels—it's solar-powered), and the caterpillar is immortal, can it ever reach the truck? Either prove that it is impossible to reach the truck, or determine how long it would take to do so.

9. **Rectangles Galore** Let S be an infinite set of rectangles; each one has one corner at $(0, 0)$ and its opposite corner at a point with positive integer coordinates. Show that there must exist rectangles $A, B \in S$ for which the interior of A is contained in the interior of B .

10. **Number Puzzle** The triangular lattice below has 8 rows and 36 circular cells. Consider all ways in which the integers $1, 2, \dots, 6$ can be filled into the lattice so that each integer appears exactly 6 times. Find the largest n for which the following statement is true: There must exist a straight line (parallel to a side of the triangle) that contains at least n different integers.



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SOLUTIONS

1. **Smallinomial** The cubic equation $ax^3 + bx^2 + cx + d = 0$ has non-zero integer coefficients and distinct integer solutions. Find the smallest possible value for $|a| + |b| + |c| + |d|$.

Solution. The smallest possible value is 6.

Let $f(x) = ax^3 + bx^2 + cx + d$, and set $S = |a| + |b| + |c| + |d|$. $S = 6$ is satisfied with $f(x) = (x + 1)(x - 1)(x - 2) = x^3 - 2x^2 - x + 2$. We need to show that the sum cannot be made smaller with some different polynomial $f(x)$.

Since $f(x)$ is required to have non-zero coefficients, $S \geq 4$. Furthermore, since $f(x)$ is required to have distinct integer solutions, $|d| \geq 2$, hence $S \geq 5$. All that remains is to show that no polynomial of the form $x^3 \pm x^2 \pm x \pm 2$ has distinct integer-valued zeroes.

To show this, first note that the zeroes would need to be 1, -1 , and one of $\{-2, 2\}$. (In order to get $|d| = 2$.) But $x^3 \pm x^2 \pm x \pm 2$ is odd for $x = \pm 1$, so these cannot be zeroes. \square

- 2. The Traveling Ant** An ant walking on the plane departs from $(0, 0)$, traveling between lattice points. From any given lattice point (x, y) , the ant randomly decides to travel to $(x + 1, y)$, $(x, y + 1)$, or $(x + 1, y + 1)$. After some time, the ant arrives at $(4, 4)$. What is the probability that the ant stopped by $(2, 2)$ along the way?

Solution. Let $p_{x,y}$ be the probability that the ant ever reaches (x, y) , and note that $p_{0,0} = 1$ and $p_{x,y} = 0$ if $x < 0$ or $y < 0$. For all other lattice points, we have $p_{x,y} = \frac{1}{3}(p_{x-1,y} + p_{x,y-1} + p_{x-1,y-1})$. From this we can build the following table:

		x				
		0	1	2	3	4
y	0	1	$\frac{1}{3^1}$	$\frac{1}{3^2}$	$\frac{1}{3^3}$	$\frac{1}{3^4}$
	1	$\frac{1}{3^1}$	$\frac{5}{3^2}$	$\frac{9}{3^3}$	$\frac{13}{3^4}$	$\frac{17}{3^5}$
	2	$\frac{1}{3^2}$	$\frac{9}{3^3}$	$\frac{33}{3^4}$	$\frac{73}{3^5}$	$\frac{129}{3^6}$
	3	$\frac{1}{3^3}$	$\frac{13}{3^4}$	$\frac{73}{3^5}$	$\frac{245}{3^6}$	$\frac{593}{3^7}$
	4	$\frac{1}{3^4}$	$\frac{17}{3^5}$	$\frac{129}{3^6}$	$\frac{593}{3^7}$	$\frac{1921}{3^8}$

The probability of ever reaching $(4, 4)$ is $p_{4,4} = \frac{1921}{3^8}$, and the probability of ever reaching both $(2, 2)$ and $(4, 4)$ is $p_{2,2}^2 = \left(\frac{33}{3^4}\right)^2 = \frac{1089}{3^8}$. Thus the probability that an ant currently at $(4, 4)$ had previously visited $(2, 2)$ is $\frac{1089}{3^8} / \frac{1921}{3^8} = \frac{1089}{1921}$. \square

- 3. Fickle Factorial** Find all integers n , $2 \leq n \leq 4010$, such that there exist integers x and y satisfying $2011! = \frac{x^2 y^3}{n!}$.

Solution. If the prime factorization of q is $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ then q can be written as $x^2 y^3$ if and only if the exponents $e_i \geq 2$ for all i . If every $e_i \geq 2$, then $e_i = 2a_i + 3b_i$ for non-negative integers a_i, b_i . Thus $x = p_1^{a_1} \cdots p_k^{a_k}$, $y = p_1^{b_1} \cdots p_k^{b_k}$ satisfies $q = x^2 y^3$. If $e_i = 1$ for some i , then there are no x, y such that $q = x^2 y^3$, since if either x or y is a multiple of p_i , then $x^2 y^3$ must be a multiple of p_i^2 .

Hence, we want to find all n for which there are no singleton primes dividing $2011!n!$. Since 2011 is prime, we must have $n \geq 2011$. The next prime after 2011 is 2017, so if $2017 \leq n \leq 4010$ then $2011!n!$ will have a single 2017 in its prime factorization. Since there are no singleton prime divisors for $2011 \leq n < 2017$, the solution set is $\boxed{\{2011, 2012, 2013, 2014, 2015, 2016\}}$. \square

Note. If we use Bertrand's Postulate – which proves there exists a prime p between n and $2n$ for all integers $n \geq 3$ – it can be shown that there are no solutions for $n \geq 4011$.

4. **Determine It** Let A be a 2011×2011 matrix whose i, j entry is $(-1)^{i+j}$, and let I_{2011} be the 2011×2011 identity matrix. Let $B(x) = A + Ix$, for $x \in \mathbb{R}$. Compute the determinant of $B(x)$.

Solution. Originally we have

$$|B(x)| = \begin{vmatrix} x+1 & -1 & +1 & -1 & \cdots & +1 \\ -1 & x+1 & -1 & +1 & \cdots & -1 \\ +1 & -1 & x+1 & -1 & \cdots & +1 \\ -1 & +1 & -1 & x+1 & \cdots & +1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ +1 & -1 & +1 & -1 & \cdots & x+1 \end{vmatrix}$$

Adding rows does not change the determinant, so add row $k+1$ to row k , for $k = 1, 2, \dots, 2010$:

$$|B(x)| = \begin{vmatrix} x & x & 0 & 0 & \cdots & 0 & 0 \\ 0 & x & x & 0 & \cdots & 0 & 0 \\ 0 & 0 & x & x & \cdots & 0 & 0 \\ 0 & 0 & 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x & x \\ +1 & -1 & +1 & -1 & \cdots & -1 & x+1 \end{vmatrix}$$

Subtracting columns does not change the determinant, so subtract row k from row $k+1$, for $k = 1, 2, \dots, 2010$:

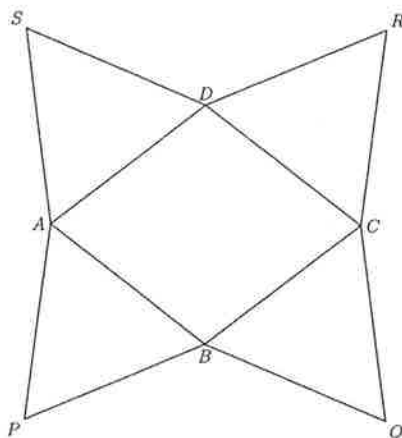
$$|B(x)| = \begin{vmatrix} x & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & x & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & x & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x & 0 \\ +1 & -2 & +3 & -4 & \cdots & -2010 & x+2011 \end{vmatrix}$$

$$= \boxed{x^{2010}(x+2011)}$$

□

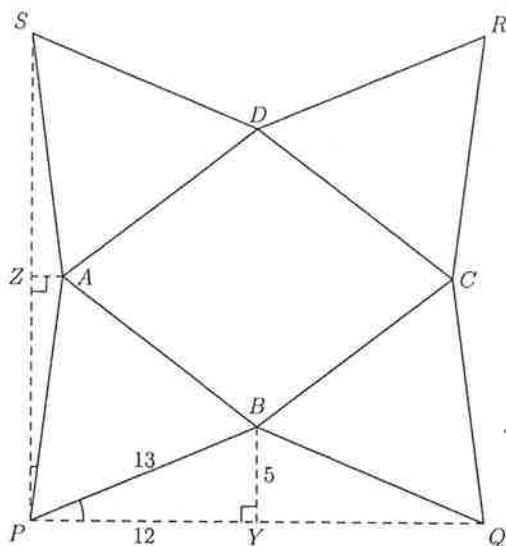
Source: International Math Competition for University Students (9th Edition)

5. **Rhomboctagon** $ABCD$ is a rhombus with side length 13. Equilateral triangles are erected on all four sides, resulting in concave octagon $APBQCRDS$. Given $PQ = 24$, compute the area of the octagon.



Solution. We will compute $[APBQCRDS]$ by computing $[PQRS]$ and subtracting the area of the 4 triangles outside $ABCD$ but inside $PQRS$.

Let Y be the midpoint of \overline{PQ} and Z be the midpoint of \overline{QS} , and let $\angle BPY = \alpha$ and $\angle APZ = \beta$. Since BPY is a right $5/12/13$ triangle, we have $\cos \alpha = \frac{12}{13}$ and $\sin \alpha = \frac{5}{13}$.



Furthermore, since $\alpha + \beta = 30^\circ$, we can use the difference of cosines and sines formulas to get $\cos \beta = \frac{12\sqrt{3}+5}{26}$, $\sin \beta = \frac{12-5\sqrt{3}}{26}$. Since $PS = 2 \cdot 13 \cos \beta = 12\sqrt{3} + 5$, we know

$[PQRS] = 24(12\sqrt{3} + 5) = 288\sqrt{3} + 120$. We also have $[BPY] = 30$, so now we need to compute $[APZ]$.

$$[APZ] = \frac{13^2}{2} \cos \beta \sin \beta = \frac{13^2}{2} \cdot \frac{12\sqrt{3} + 5}{26} \cdot \frac{12 - 5\sqrt{3}}{26} = \frac{119\sqrt{3} - 120}{8}.$$

Putting it all together:

$$\begin{aligned} [APBQCRDS] &= [PQRS] - 4 \cdot [BPY] - 4 \cdot [APZ] \\ &= (288\sqrt{3} + 120) - 4 \cdot 30 - 4 \cdot \left(\frac{119\sqrt{3} - 120}{8} \right) \\ &= 60 + \frac{457\sqrt{3}}{2} \end{aligned}$$

6. **Integral** Compute the exact value of $\int_0^{\pi/2} \ln \cos x \, dx$.

Solution. Let $I = \int_0^{\pi/2} \ln \cos x \, dx$, and note that if we substitute $u = \frac{\pi}{2} - x$ we get

$I = \int_0^{\pi/2} \ln \sin x \, dx$. By symmetry, $2I = \int_0^{\pi} \ln \sin x \, dx$, from which we get

$$\begin{aligned} 2I &= \int_0^{\pi} \ln \sin x \, dx \\ &= \int_0^{\pi} \ln \sin \left(2 \cdot \frac{x}{2} \right) \, dx \\ &= \int_0^{\pi} \ln \left(2 \sin \frac{x}{2} \cos \frac{x}{2} \right) \, dx \\ &= \int_0^{\pi} \ln 2 + \ln \sin \frac{x}{2} + \ln \cos \frac{x}{2} \, dx \\ &= \pi \ln 2 + 2 \left(\int_0^{\pi/2} \ln \sin u + \ln \cos u \, du \right) \\ &= \pi \ln 2 + 4I, \end{aligned}$$

from which we get $I = -\frac{\pi \ln 2}{2}$

□

7. **Stay Positive** Show that there are no positive integer solutions to

$$a^4 + b^4 + c^4 + d^4 + e^4 = 16abcde.$$

Solution. The residues of $x^4 \pmod{16}$ are 0, 1, and 9, so the only way for the lefthand side to be a multiple of 16 is if a, b, c, d, e are all even. Let $(a, b, c, d, e) = (2^k u, 2^k v, 2^k w, 2^k x, 2^k y)$, where at least one of u, v, w, x, y is odd. Then the largest power of 2 dividing the lefthand side is no larger than 2^{4k+2} , obtainable if exactly 4 of u, v, w, x, y are odd. But the righthand side is divisible by 2^{5k+4} . Thus, no solutions are possible. \square

8. **Stretch Yourself** An infinitely stretchable rubber band connects the rear of a truck to a post. At first, the rubber band stretches 1 meter. A caterpillar starts walking along the rubber band, at a constant speed of 1 centimeter per second. At that same moment, the truck drives away from the post, at a constant speed of 100 meters per second. As the truck speeds away, the rubber band stretches uniformly. If the band never breaks, the truck never stops (or refuels – it’s solar-powered), and the caterpillar is in good health, can it ever reach the truck? Either prove that it is impossible to reach the truck, or determine how long it would take to do so.

Solution. Let $d(t)$ denote the distance the caterpillar is from the post (in meters) at time t . Let $f(t)$ be the *fraction* of the way along the rubber band the caterpillar has traveled at time t . Let $\ell(t)$ equal the length of the rubber band at time t . We have the following system of equations:

$$\begin{aligned} d(0) &= f(0) = 0 \\ \ell(t) &= 1 + 100t \\ f(t) &= \frac{d(t)}{\ell(t)} \\ d'(t) &= .01 + 100f(t) \end{aligned}$$

The last equation describes that the caterpillar’s progress (with respect to the post) increases linearly with its fractional progress along the rubber band, plus its own walking.

From the third equation, we have $d'(t) = 100f(t) + f'(t)(1 + 100t)$. Setting this equal to the fourth equation, we get

$$\begin{aligned} 100f(t) + f'(t)(1 + 100t) &= .01 + 100f(t) \\ f'(t) &= \frac{.01}{1 + 100t} \\ f(t) &= \int \frac{dt}{100 + 10000t} \\ &= \frac{\ln(100 + 10000t)}{10000} + C. \end{aligned}$$

Since $f(0) = 0$, we get $C = -\frac{\ln(100)}{10000}$. Solving $f(t) = 1$, we get $t = \frac{e^{10000}-1}{100}$. So **yes**, the caterpillar will reach the truck, but not until long after the earth crashes into the sun. \square

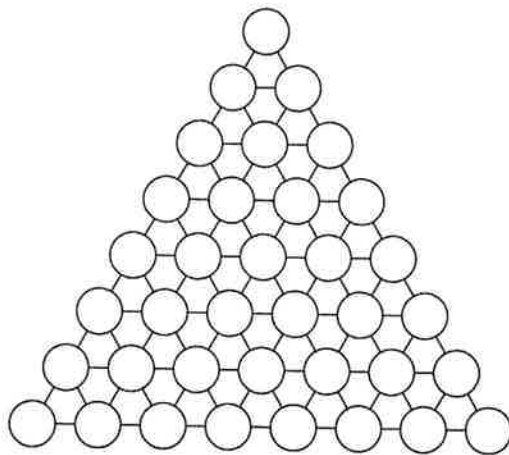
- 9. Rectangles Galore** Let S be an infinite set of rectangles that have one corner at $(0,0)$ and their opposite corner at a point with positive integer coordinates. Show that there must exist rectangles $A, B \in S$ for which the interior of A is contained in the interior of B .

Solution. Suppose no pair of boxes exists. For a box $A \in S$, let $P(A)$ be the corner of A opposite the origin. Suppose box A is the box for which $P(A)$ has a minimal x -coordinate. Then the y -coordinate of $P(A)$ must be maximal over S , and from here we again build a chain of boxes A, A_1, A_2, \dots such that the x -coordinates of $P(A_n)$ are strictly increasing, and the y -coordinates are strictly decreasing, which contradicts the condition that $|S| = \infty$.

****Note:** This problem was intended to use rectangles whose sides are parallel to the x and y -axes. This was not stated in the given problem, so there does exist an infinite set of rectangles for which the interior of no rectangle is contained in the interior of any other rectangle. For example, consider the set of rectangles whose corners are at $(0,0), (n,1), (n-1, n+1), (-1, n)$ for all integers $n > 0$.

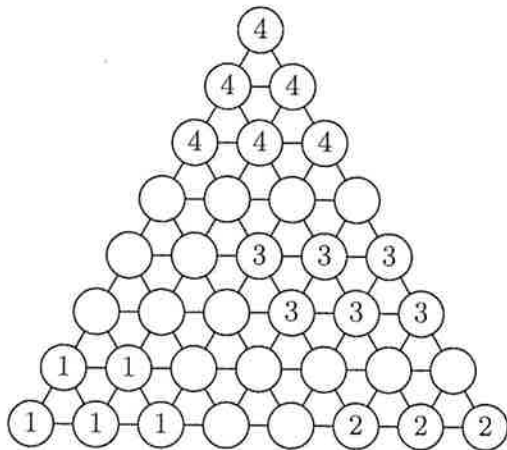
Source: Brazilian Math Olympiad

- 10. Number Puzzle** The triangular lattice below contains 8 rows and 36 cells. Consider all ways in which the integers $1, 2, \dots, 6$ can be filled into the lattice so that each integer appears exactly 6 times. Find the largest n for which the following statement is true: There must exist a straight line (parallel to a side of the triangle) that contains at least n different integers.



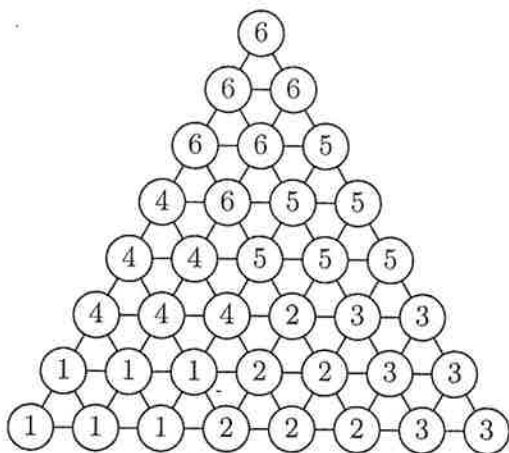
Solution. The largest n is $n = 3$. We will first show n is at least 3 for any arrangement, then we will provide an example to show that $n = 3$ is achievable.

Consider how we place the six 1's, 2's, etc. Let NW_k , NE_k , and EW_k equal the number of north-to-west, north-to-east, and east-to-west lines that contain a k , for $k \in \{1, 2, 3, 4, 5, 6\}$. We claim that $NW_k + NE_k + EW_k \geq 9$. To show this, consider the largest number of k 's that can be placed so that $NW_k + NE_k + EW_k \leq 8$. We would need one of the numbers – say EW_k – to be at most 2. If $EW_k = 2$, we can only place at most 5 k 's. (See the 1's, below.) If $EW_k = 1$, we can only place 3. (See the 2's.)



However, it is possible for $NW_k + NE_k + EW_k = 9$, as the 3's and 4's show two such arrangements. (Note that we are counting, in each direction, the “singleton” lines that only contain a corner circle.)

Since $\sum_{k=1}^6 NW_k + NE_k + EW_k \geq 54$, one of the 24 lines must contain at least 3 different numbers. Hence, $n \geq 3$. We finish by demonstrating that $n = 3$ is possible:



□