Twentyfirst Annual Konhauser Problem Fest
Problems by Răzvan Gelca

Calculators of any sort are allowed, although complete justifications are expected, not just the statement of the answer. Use of cell-phones and computers is not permitted. Partial credit will be given for progress toward a solution or the answer to part of a question.

1. Let n be a positive integer. In how many ways can the numbers $1, 2, 3, \ldots, n^2$ be arranged in an $n \times n$ array a_{ij}, $i, j = 1, 2, \ldots, n$, so that for every i, $a_{i1}, a_{i2}, \ldots, a_{in}$ form an arithmetic sequence and for every j, $a_{1j}, a_{2j}, \ldots, a_{nj}$ form an arithmetic sequence?

2. The angles of a certain triangle are measured in radians and the product of these measures is equal to $\pi^3/30$. Prove that the triangle is acute.

3. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$(f \circ f \circ f)(x) = x^3 \text{ and } (f \circ f \circ f \circ f \circ f)(x) = x^5,$$

for every $x \in \mathbb{R}$?

4. Compute the integral

$$\iint_D \frac{2x^6 - 3y^6}{x^6 + y^6} dxdy$$

where D is the region in the plane that lies between the square with vertices $(3, 0)$, $(0, 3)$, $(-3, 0)$, $(0, -3)$ and the square with vertices $(1, 1)$, $(-1, 1)$, $(-1, -1)$, $(1, -1)$.

5. Define the sequence

$$x_n = \sum_{k=1}^{n} \frac{k}{n^2 + 2k^2}, \quad n \geq 1.$$

Prove that the sequence x_n converges and find its limit.

6. Let $f_0(x) = x$, $g_0(x) = x$ and for $n \geq 0$,

$$f_{n+1}(x) = \ln[1 + 3(f_n(x))^2], \quad g_{n+1}(x) = \ln[1 + 5(g_n(x))^2].$$

Prove that the limit

$$\lim_{x \to 0} \frac{f_{2014}(x)}{g_{2014}(x)}$$

exists and compute it.

7. Find all real numbers x and y that are solutions to the system of equations

$$\begin{aligned} 3^x - 3^y &= 2^y \\ 9^x - 6^y &= 19^y. \end{aligned}$$

8. Let a and b be integers such that $a + b = 2014$. Prove that the determinant

$$\begin{vmatrix} a^3 & b^3 & 3ab & -1 \\ -1 & a^2 & b^2 & 2ab \\ 2b & -1 & a^2 & -b^2 \\ 0 & b & -1 & a \end{vmatrix}$$

is a multiple of 61.

9. Let p be an odd prime. Show that the number

$$1^p + 2^p + \cdots + p^p$$

is divisible by p^2 but not by p^3.

10. Let ABC and DAB be right isosceles triangles such that $\angle A = \angle D = 90^\circ$, $AB = 1$, and C and D are separated by the line AB. Let M be a point on the segment AC, N the intersection of DM with BC and P the intersection of BM with AN. Show that when M varies on the side AC then P describes a smooth arc, and find the length of this arc.

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Solutions:

1. Let n be a positive integer. In how many ways can the numbers $1, 2, 3, \ldots, n^2$ be arranged in an $n \times n$ array a_{ij}, $i, j = 1, 2, \ldots, n$, so that for every i, $a_{i1}, a_{i2}, \ldots, a_{in}$ form an arithmetic sequence and for every j, $a_{1j}, a_{2j}, \ldots, a_{nj}$ form an arithmetic sequence?

Solution: The answer is 8.

Because there are no non-positive numbers in the array, 1 has to be in one corner. Reflections of the array in the vertical or horizontal preserve the properties of rows and columns to be arithmetic, so it suffices to treat the case where 1 is in the upper left corner. The next smallest element is 2, and it can only lie next to 1. If 2 lies on the same row as 1 then the first row is $1, 2, \ldots, n$.

The next smallest number is $n+1$, and it can only lie right below 1, or else it lies either horizontally or vertically between two numbers larger than it. So the first column is $1, n+1, \ldots, n^2-n+1$.

The next smallest number is $n+2$, and since it cannot lie between two larger numbers horizontally or vertically we must have $a_{22} = n+2$. Then the array can only be completed uniquely as $a_{ij} = (n-1)i + j$.

In the other situation 2 is right below and then array is the reflection of this one over the main diagonal. Placing 1 in the other 3 corners, we obtain $2 \cdot 3 = 6$ more possibilities. The problem is solved.

2. The angles of a certain triangle are measured in radians and the product of these measures is equal to $\pi^3/30$. Prove that the triangle is acute.

Solution: We will show that if a triangle is right or obtuse, then the product of the measures of its angles is at most $\pi^3/32$. Let x, y, z be these measures, and assume that x is obtuse. We consider the domain

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid \pi/2 \le x \le \pi, y \ge 0, z \ge 0, x + y + z = \pi\}$$

and the function

$$f : D \to [0, \infty), \quad f(x, y, z) = xyz.$$

The method of Lagrange multipliers shows that the maxima of f are reached either on the boundary of the planar domain D, or at the points (x, y, z) which arise by solving the system of equations

$$\begin{aligned} xy &= \lambda \\ yz &= \lambda \\ zx &= \lambda \\ x + y + z &= \pi. \end{aligned}$$

The only solution to this system is $(\pi/3, \pi/3, \pi/3)$, which is not in the domain. Hence we must examine the boundary. The part of the boundary with $y = 0$ or $z = 0$ yields a minimum for the function, so we only focus on the part where $x = \pi/2$. In this case f becomes the two variable function $f(\pi/2, y, z) = \pi/2yz$, which, by using the constraint, is turned into a quadratic

$$f(\pi/2, y, \pi/2 - y) = \pi/2y(\pi/2 - y).$$

The maximum of its quadratic is at its vertex, and is equal to $\pi^3/32$. We conclude that our triangle cannot be right or obtuse, so it is acute.

3. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$(f \circ f \circ f)(x) = x^3 \text{ and } (f \circ f \circ f \circ f \circ f)(x) = x^5,$$

for every $x \in \mathbb{R}$?

Solution: If f is such a function, then

$$(f \circ f \circ f \circ f \circ f)(x) = [(f \circ f) \circ (f \circ f \circ f)](x) = (f \circ f)(x^3) = x^5.$$

So

$$(f \circ f)(x) = x^{5/3}.$$

Then

$$(f \circ f \circ f)(x) = [f \circ (f \circ f)](x) = f(x^{5/3}) = x^3.$$

Hence

$$f(x) = x^{9/5}.$$

But this function does not satisfy either of the conditions from the statement. Hence the answer is negative.

4. Compute the integral

$$\iint_D \frac{2x^6 - 3y^6}{x^6 + y^6} dxdy$$

where D is the region in the plane that lies between the square with vertices $(3,0)$, $(0,3)$, $(-3,0)$, $(0,-3)$ and the square with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$.

Solution: Note that D is invariant under the change of coordinates $x \mapsto y, y \mapsto x$. Hence

$$\iint_D \frac{x^6}{x^6 + y^6} dxdy = \iint_D \frac{y^6}{x^6 + y^6} dxdy$$

and so

$$\iint_D \frac{x^6}{x^6 + y^6} dxdy = \frac{1}{2} \iint_D \frac{x^6 + y^6}{x^6 + y^6} dxdy = \iint_D dxdy.$$

The last integral is the area of D, which is $18 - 4 = 14$. We conclude that the integral from the statement is equal to

$$2 \iint_D \frac{x^6}{x^6 + y^6} dxdy - 3 \iint_D \frac{y^6}{x^6 + y^6} dxdy = 2 \cdot \frac{1}{2} \cdot 14 - 3 \cdot \frac{1}{2} \cdot 14 = -7.$$

5. Define the sequence

$$x_n = \sum_{k=1}^{n} \frac{k}{n^2 + 2k^2}, \quad n \geq 1.$$

Prove that the sequence x_n converges and find its limit.

Solution: Rewrite the formula for the term of the sequence as

$$x_n = \sum_{k=1}^{n} \frac{\frac{k}{n}}{1 + 2\left(\frac{k}{n}\right)^2} \cdot \frac{1}{n}.$$

We recognize a Riemann sum for the integral

$$\int_0^1 \frac{x}{1+2x^2}dx.$$

Hence the sequence converges to the value of this integral, which is $\frac{1}{4}\ln 2$.

6. Let $f_0(x) = x$, $g_0(x) = x$ and for $n \geq 0$,

$$f_{n+1}(x) = \ln[1 + 3(f_n(x))^2], \quad g_{n+1}(x) = \ln[1 + 5(g_n(x))^2].$$

Prove that the limit

$$\lim_{x\to 0} \frac{f_{2014}(x)}{g_{2014}(x)}$$

exists and compute it.

Solution: We want to find a formula for $\lim_{x\to 0} f_n(x)/g_n(x)$. Inductively we prove that

$$\lim_{x\to 0} f_n(x) = \lim_{n\to 0} g_n(x) = 0,$$

so we might be able to apply l'Hospital.

Let us check small cases of n. For $n = 1$, l'Hospital can indeed be applied, since

$$\lim_{x\to 0} \frac{f_1'(x)}{g_1'(x)} = \lim_{x\to 0} \frac{\frac{1}{1+3x^2} \cdot 2 \cdot 3x}{\frac{1}{1+5x^2} \cdot 2 \cdot 5x} = \frac{3}{5}.$$

So $\lim_{x\to 0} f_1(x)/g_1(x) = 3/5$. Also for $n = 2$, we check

$$\lim_{x\to 0} \frac{f_2'(x)}{g_2'(x)} = \lim_{x\to 0} \frac{\frac{1}{1+2(f_1(x))^2} \cdot 2 \cdot 3 f_1(x) f_1'(x)}{\frac{1}{1+3g_1(x)} \cdot 2 \cdot 5 \cdot g_1(x) g_1'(x)} = \frac{3^3}{5^3}.$$

So $\lim_{x\to 0} f_1(x)/g_1(x) = 3^3/5^3$. Inductively we prove that

$$\lim_{x\to 0} \frac{f_n(x)}{g_n(x)} = \lim_{x\to 0} \frac{f_n'(x)}{g_n'(x)} = \frac{3^{2^n-1}}{5^{2^n-1}}.$$

Indeed,

$$\begin{aligned}
\lim_{n\to 0}\frac{f'_{n+1}(x)}{g'_{n+1}(x)} &= \lim_{n\to 0}\frac{\frac{1}{1+3(f_n(x))^2}\cdot 2\cdot 3\cdot f_n(x)f'_n(x)}{\frac{1}{1+5(g_n(x))^2}\cdot 2\cdot 5g_n(x)g'_n(x)}\\
&= \frac{3}{5}\lim_{x\to 0}\frac{1+5(g_n(x))^2}{1+3(f_n(x))^2}\cdot\lim_{x\to 0}\frac{f_n(x)}{g_n(x)}\cdot\lim_{x\to 0}\frac{f'_n(x)}{g'_n(x)}\\
&= \frac{3}{5}\cdot 1\cdot\frac{3^{2^n-1}}{5^{2^n-1}}\cdot\frac{3^{2^n-1}}{5^{2^n-1}} = \frac{3^{2^n+1}}{5^{2^n+1}}.
\end{aligned}$$

Hence the answer to the problem is

$$\lim_{x\to 0}\frac{f_{2014}(x)}{g_{2014}(x)} = \frac{3^{2^{2014}-1}}{5^{2^{2014}-1}}.$$

7. Find all real numbers x and y that are solutions to the system of equations

$$\begin{aligned}
3^x - 3^y &= 2^y\\
9^x - 6^y &= 19^y.
\end{aligned}$$

Solution: Write the system as

$$\begin{aligned}
3^x &= 2^y + 3^y\\
9^x &= 6^y + 19^y.
\end{aligned}$$

Square the first equation then substitute 9^x from the second to obtain

$$4^y + 2\cdot 6^y + 9^y = 6^y + 19^y.$$

Rewrite this as

$$4^y + 6^y + 9^y = 19^y.$$

It is easy to see that $y = 1$ is a solution. There are no other solutions to this equation because after dividing by 19^y we obtain

$$\left(\frac{4}{19}\right)^y + \left(\frac{6}{19}\right)^y + \left(\frac{9}{19}\right)^y = 1$$

and the left-hand side is a strictly decreasing function which assumes the value 1 exactly once.

We conclude that the only pair of real numbers satisfying the system is $(\log_3 5, 1)$.

8. Let a and b be integers such that $a+b = 2014$. Prove that the determinant

$$\begin{vmatrix} a^3 & b^3 & 3ab & -1 \\ -1 & a^2 & b^2 & 2ab \\ 2b & -1 & a^2 & -b^2 \\ 0 & b & -1 & a \end{vmatrix}$$

is a multiple of 61.

Solution: Add the second, third, and fourth to the first. By using the Cremona identity

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-xz)$$

we can write

$$\begin{aligned} a^3 + b^3 - 1 + 3ab &= a^3 + b^3 + (-1)^3 - 3ab(-1) \\ &= (a+b-1)(a^2+b^2+1-ab+a+b). \end{aligned}$$

Also

$$-1 + a^2 + b^2 + 2ab = (a+b)^2 - 1 = (a+b-1)(a+b+1)$$

and

$$2b - 1 + a^2 - b^2 = a^2 - (b-1)^2 = (a+b-1)(a-b+1).$$

It follows that the determinant is divisible by $a+b-1 = 3 \times 11 \times 61$.

9. Let p be an odd prime. Show that the number

$$1^p + 2^p + \cdots + p^p$$

is divisible by p^2 but not by p^3.

Solution: We can ignore the last term since it is divisible by p^3. Because p is prime we can group the other terms in pairs $k^p + (p-k)^p$ and write

$$\begin{aligned} k^p + (p-k)^p = k^p + p^p - \binom{p}{1} p^{p-1} + \cdots - \binom{p}{p-2} p^2 k^{p-2} \\ + \binom{p}{p-1} p k^{p-1} - k^p = p^3 M + p^2 k^{p-1} \end{aligned}$$

where M is an integer. Here we used the fact that $\binom{p}{p-2} = p(p-1)/2$ is a multiple of p. On the other hand, by Fermat's Little Theorem, $k^{p-1} \equiv 1 (\mathrm{mod}\ p)$. So this is of the form $p^3N + p^2$. Adding all these sums for $k = 1, 2, \ldots, \frac{p-1}{2}$, we obtain that the expression from the statement is a multiple of p^3 plus $\frac{p-1}{2}p^2$, and we are done.

10. Let ABC and DAB be right isosceles triangles such that $\angle A = \angle D = 90°$, $AB = 1$, and C and D are separated by the line AB. Let M be a point on the segment AC, N the intersection of DM with BC and P the intersection of BM with AN. Show that when M varies on the side AC then P describes a smooth arc, and find the length of this arc.

Solution: Choosing a Cartesian system of coordinates with origin at A and axes AB and AC, we have $A(0, 0)$, $B(1, 0)$, $C(0, 1)$, $D(1/2, -1/2)$. Let $M(0, t)$, $t \in [0, 1]$. Then

$$\begin{aligned} BC:&\quad y = -x + 1 \\ DM:&\quad y = -(2t + 1)x + t. \end{aligned}$$

Hence $N(\frac{t-1}{2t}, \frac{t+1}{2t})$. We have

$$\begin{aligned} BM:&\quad y = -tx + t \\ AN:&\quad y = \frac{t+1}{t-1}x, \end{aligned}$$

and so

$$P\left(\frac{t^2 - t}{t^2 + 1}, \frac{t^2 + t}{t^2 + 1}\right).$$

Let us find the cartesian equation of the arc that P describes. We want to eliminate t from the equations $x = (t^2 - t)/(t^2 + 1)$ and $y = (t^2 + t)/(t^2 + 1)$. We have

$$\begin{aligned} t^2 - t &= (t^2 + 1)x \\ t^2 + t &= (t^2 + 1)y. \end{aligned}$$

Adding and subtracting we get

$$\begin{aligned} 2t^2 &= (t^2 + 1)(x + y) \\ 2t &= (t^2 + 1)(x - y). \end{aligned}$$

Dividing we obtain

$$t = \frac{x+y}{x-y}.$$

After substituting in $2t = (t^2+1)(x-y)$ and performing the algebraic computations we obtain

$$x^2 + y^2 - x - y = 0,$$

which is the equation of the circle of radius $\sqrt{2}/2$ centered at $(1/2, 1/2)$. The arc of curve in question is the arc of this circle with endpoints A and C; its length is $1/4$ of the total circle, hence $\pi\sqrt{2}/4$.