Twenty-third Annual Konhauser Problemfest

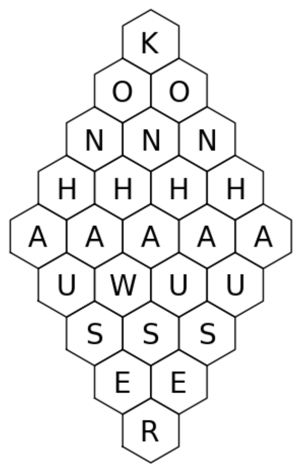
St. Olaf College **February 28, 2015**

Problems set by David Molnar, Rutgers University

INSTRUCTIONS: Each team must hand in all work to be graded at the same time (at the end of the three-hour period). Each problem must be written on a separate page (or pages) and YOUR TEAM NAME SHOULD APPEAR AT THE TOP OF EVERY PAGE. Only one version of each problem will be accepted per team. Calculators of any sort are allowed. Justifications and/or explanations are expected for all problems. All ten problems will be weighted equally, and partial credit will be given for substantial progress toward a solution.

1. "U" Substitution

How many ways are there to spell KONHAUSER in the grid below, moving between adjacent hexagons from top to bottom?



2. Partial Credit?

A student is given two values of a function — $f(3)$ and $f(7)$ — and asked to find $f(5)$. The function is to be an exponential function, but the student uses a linear function instead, and as a result his answer is exactly one away from the correct answer. The only mistake the student makes is using the wrong type of function. If $f(3)$ is 98, what could $f(7)$ have been? There are two possible values; give their **sum**.

3. Convex Dissection

Let A be the annulus $\{(x, y) \mid 1 \leq x^2 + y^2 \leq 4\}$. Find, with proof, the smallest number of convex sets into which A can be partitioned, or prove that no such number exists.
(Recall that a set S in the plane is *convex* if, for any points $\mathbf{p}$ and $\mathbf{q}$ in S, the line segment between $\mathbf{p}$ and $\mathbf{q}$ lies entirely within S; that is, if $t\mathbf{p}+(1-t)\mathbf{q} \in S$ for all $t \in [0, 1]$.)

4. An Army of Logs

Let $f(x) = \ln(1 + x)$, and define an infinite sequence by $x_1 = 1$, $x_n = f(x_{n-1})$ for all $n \geq 2$. Determine, with proof, the convergence or divergence of the series $\sum_{n=1}^{\infty} x_n$.

5. A Fractal Integral

Let r be the unique continuous function on $[0,1]$ satisfying

$$r(x)=\begin{cases}\frac{1}{3}r(3x) & \text{if } 0\le x<\frac{1}{3}\\ \frac{1}{6}-|x-\frac{1}{2}| & \text{if } \frac{1}{3}\le x\le\frac{2}{3}\\ \frac{1}{3}r(3x-2) & \text{if } \frac{2}{3}<x\le 1.\end{cases}$$

Determine $\int_0^1 r(x)\,dx$.

6. Circle Limit III

Let C_1 denote the circle of radius 1 centered at $(0,1)$. Let C_2 be the circle tangent to the x-axis at $x=c$ for arbitrary $c>0$, and to C_1. Let C_3 be the circle tangent to C_1 and C_2, and to the x-axis at a point $x=d$, $0<d<c$. If $r(c)$ is the radius of C_2 and $s(c)$ is the radius of C_3, calculate $\lim_{c\to 0} s(c)/r(c)$.

7. A Colorful Conflict

Let S be the collection of positive integers that are divisors of 27,000 but not of 900. We will say that two elements of S are *compatible* if one is a multiple of the other.

Archimedes and Beethoven are playing a game in which they assign colors to elements of S. A number can be colored red if it is a multiple of 2, blue if it is a multiple of 3, and green if it is a multiple of 5. A player colors *at least one* number on his or her turn, and once a number has been assigned a color, that color may not be changed. If, after a player's turn, there are three numbers, pairwise compatible, with three different colors, then that player loses.

Archimedes plays first. Describe a winning strategy for one of the two players.

8. All or Nothing

Prove that every (real-valued) solution of the differential equation $y'''=y$ has either no real roots, or infinitely many.

9. Find the Sides

A right triangle T has integer side lengths, none greater than 400, with no common factor. If the average distance from a point on the interior of T to its boundary is 6, what are the lengths of the sides of T?

10. Konhauser Caravan

Legend has it that the first Konhauser Problemfest was carried through the desert by four camels named Mac, Carl, Ole, and Tom. Their journey was long and difficult, but they were aided by the presence of three oases along their route, so Mac carried the problems until the first oasis, Carl from the first to the second, Ole from the second to the third, and Tom the rest of the way. Exact details have been buried by the sands of time, so we no longer know how the oases were positioned. Assuming an equal likelihood of an oasis at any point along the route, what is the probability that Carl carried the load further than Mac, and Tom carried the load further than Carl, but not as far as the other three camels combined?

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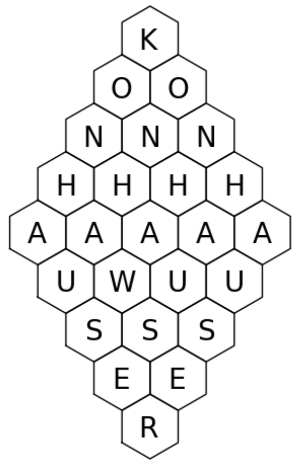
Solutions

1. How many ways are there to spell KONHAUSER in the grid below, moving between adjacent hexagons from top to bottom?



Solution: There are **40** ways to spell KONHAUSER. Any path from the K to the R consists of eight steps – four down and to the left, and four down and to the right. We can represent any such path by a string of four Ls and four Rs, such as LRRLLRLR. There are $\frac{8!}{4!4!} = 70$ such paths, but some of these, including the one given, spell KONHAWSER. A path from the K to the W consists of two left steps and three right steps; there are $\frac{5!}{2!3!} = 10$ of these. There are also three paths from the W to the R. So out of the 70 paths, $10 \cdot 3 = 30$ spell KONHAWSER. Therefore there are $70 - 30 = 40$ ways to spell KONHAUSER.

2. A student is given two values of a function — $f(3)$ and $f(7)$ — and asked to find $f(5)$. The function is to be an exponential function, but the student uses a linear function instead, and as a result his answer is exactly one away from the correct answer. The only mistake the student makes is using the wrong type of function. If $f(3)$ is 98, what could $f(7)$ have been? There are two possible values; give their **sum**.

Solution: Write $a = f(3)$ and $b = f(7)$. Since 5 is the midpoint between 3 and 7, if f is a linear function, $f(5)$ is the arithmetic mean of a and b, and if f is an exponential function, $f(5)$ is their geometric mean. The arithmetic mean of a and b, $\frac{a+b}{2}$ is always greater than or equal to the geometric mean, $\sqrt{ab}$, so if the student's answer was off by 1, we have $\frac{a+b}{2} = \sqrt{ab} + 1$. Taking $a = 98$, we have

$$\begin{aligned} \frac{98+b}{2} &= \sqrt{98b} + 1 \\ 48 + \frac{b}{2} &= \sqrt{98b} \\ 2304 + 48b + \frac{b^2}{4} &= 98b \\ b^2 - 200b + 9216 &= 0 \end{aligned}$$

Now, you are welcome to factor that if you want, but note that we are asked to find the **sum** of the two possible values, and for a monic quadratic, the sum of the roots is the opposite of the coefficient of x. So the answer is **200**. (Indeed, the two possible values of $f(7)$ are 72 and 128.)

3. Let A be the annulus $\{(x,y) \mid 1 \le x^2+y^2 \le 4\}$. Find, with proof, the smallest number of convex sets into which A can be partitioned, or prove that no such number exists.
(Recall that a set S in the plane is *convex* if, for any points $\mathbf{p}$ and $\mathbf{q}$ in S, the line segment between $\mathbf{p}$ and $\mathbf{q}$ lies entirely within S; that is, if $t\mathbf{p}+(1-t)\mathbf{q} \in S$ for all $t \in [0,1]$.)

Solution: There is no such number. Assume, to the contrary, that there is a partition of the annulus into n regions $R_1 \dots R_n$, such that each R_j is convex, $1 \le j \le n$. These regions will be our pigeonholes. For the pigeons, take all the points along the inner boundary of the annulus, namely $\{(\cos(\theta),\sin(\theta)) \mid 0 \le \theta < 2\pi\}$. There are infinitely many such points, so by the Pigeonhole Principle, there are some two pigeons in the same R_j. Let us call these two points $(\cos(\theta_1),\sin(\theta_1))$ and $(\cos(\theta_2),\sin(\theta_2))$. Since R_j is convex, the midpoint between these two points must also be in R_j. This midpoint is $\left(\frac{\cos(\theta_1)+\cos(\theta_2)}{2}, \frac{\sin(\theta_1)+\sin(\theta_2)}{2}\right)$. The distance of this point from the origin is

$$\begin{aligned}
&\sqrt{\left(\frac{\cos(\theta_1)+\cos(\theta_2)}{2}\right)^2+\left(\frac{\sin(\theta_1)+\sin(\theta_2)}{2}\right)^2}\\
&=\sqrt{\frac{\cos^2(\theta_1)+\sin^2(\theta_1)+\cos^2(\theta_2)+\sin^2(\theta_2)+2\cos(\theta_1)\cos(\theta_2)+2\sin(\theta_1)\sin(\theta_2)}{4}}\\
&=\tfrac{1}{2}\sqrt{2+2\left(\cos(\theta_1)\cos(-\theta_2)-\sin(\theta_1)\sin(-\theta_2)\right)}\\
&=\tfrac{1}{2}\sqrt{2+2\cos(\theta_1-\theta_2)}.
\end{aligned}$$

Since $\theta_1 \ne \theta_2$, $\cos(\theta_1-\theta_2)$ is strictly less than 1, so the quantity under the radical is less than 4. Therefore the point $\left(\frac{\cos(\theta_1)+\cos(\theta_2)}{2}, \frac{\sin(\theta_1)+\sin(\theta_2)}{2}\right)$ is distance less than 1 from the origin, which means it is not in A! Consequently it is not in R_j either, which is a contradiction. Thus our original assumption, that such a partition exists, is false.

4. Let $f(x) = \ln(1+x)$, and define an infinite sequence by $x_1 = 1$, $x_n = f(x_{n-1})$ for all $n \ge 2$. Determine, with proof, the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} x_n$.

Solution: The series **diverges**. This is seen by direct comparison to the harmonic series. We claim that $x_n \ge \dfrac{1}{n}$ for all n. This is shown by induction. The inequality is trivial for $n=1$. Recall that the function $f(x) = \ln(1+x)$ has Taylor series expansion $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dots$, the radius of convergence of which is 1. Thus, for any x, $-1 < x < 1$, we have the inequality $f(x) > x - \dfrac{x^2}{2}$. For any $n \ge 1$, we have $x_{n+1} = f(x_n) = \ln(1+x_n)$. Assume, by the induction hypothesis, that $x_n \ge \dfrac{1}{n}$. Since the logarithm is an increasing function, $x_{n+1} \ge \ln\left(1+\dfrac{1}{n}\right)$. Using the inequality from the Taylor series yields

$$x_{n+1} \ge \ln\left(1+\frac{1}{n}\right) \ge \frac{1}{n} - \frac{1}{2n^2} \ge \frac{1}{n} - \frac{1}{n^2+n} = \frac{1}{n+1}.$$

This completes the proof by induction; since $\sum_{n=1}^{\infty} x_n$ is greater than or equal to the harmonic series term-by-term, as the harmonic series diverges, $\sum_{n=1}^{\infty} x_n$ does as well.

5. Let r be the unique continuous function on $[0, 1]$ satisfying

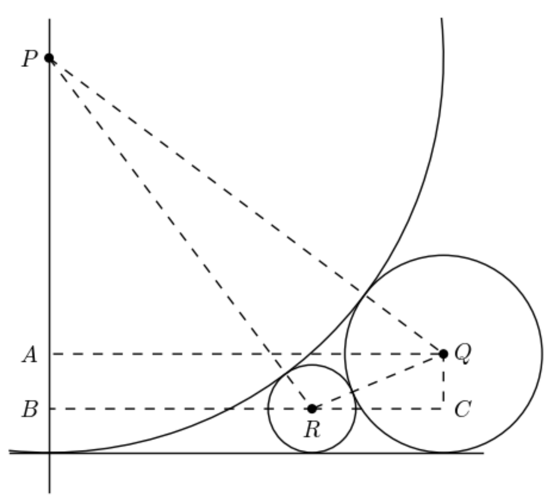
$$r(x) = \begin{cases} \frac{1}{3}r(3x) & \text{if } 0 \le x < \frac{1}{3} \\ \frac{1}{6} - |x - \frac{1}{2}| & \text{if } \frac{1}{3} \le x \le \frac{2}{3} \\ \frac{1}{3}r(3x-2) & \text{if } \frac{2}{3} < x \le 1. \end{cases}$$

Determine $\int_0^1 r(x)\,dx$.

Solution: Write $I = \int_0^1 r(x)\,dx$. Split the integral up as $\int_0^{1/3} r(x)\,dx + \int_{1/3}^{2/3} r(x)\,dx + \int_{2/3}^{1} r(x)\,dx$. The middle integral is equal to $\int_{1/3}^{2/3} \left(\frac{1}{6} - \left|x - \frac{1}{2}\right|\right) dx$, which is the area of a triangle with base $\frac{1}{3}$ and height $\frac{1}{6}$, that is, $1/36$. Rewrite the first integral as $\int_0^{1/3} \frac{1}{3} r(3x)\,dx$, then use the substitution $u = 3x$, yielding $\frac{1}{9}\int_0^1 r(u)\,du = \frac{1}{9}I$. Similarly, the substitution $u = 3x - 2$ yields $\int_{2/3}^{1} r(x)\,dx = \frac{1}{9}I$. Thus, $I = \frac{1}{9}I + \frac{1}{36} + \frac{1}{9}I$. Then $\frac{7}{9}I = \frac{1}{36}$, so $I = 1/28$.

6. Let C_1 denote the circle of radius 1 centered at $(0, 1)$. Let C_2 be the circle tangent to the x-axis at $x = c$ for arbitrary $c > 0$, and to C_1. Let C_3 be the circle tangent to the x-axis and to C_1 and C_2. If $r(c)$ is the radius of C_2 and $s(c)$ is the radius of C_3, calculate $\lim_{c\to 0} s(c)/r(c)$.

Solution: As in the diagram below, let P, Q, and R denote the centers of C_1, C_2, and C_3 respectively. Also, let $A = (0, r)$, $B = (0, s)$, and $C = (c, s)$. From the right triangle ΔPAQ we obtain the relation $(1 - r)^2 + c^2 = (1 + r)^2$. Solving for r, we have $r = c^2/4$. Applying the same reasoning to ΔPBR yields $s = d^2/4$.



We still need to determine s in terms of c. Using the right triangle ΔQCR, we have $(r-s)^2+(c-d)^2=(r+s)^2$. This simplifies to $(c-d)^2=4rs$, but we can substitute in $r=c^2/4$ and $s=d^2/4$ to obtain $(c-d)^2=c^2d^2/4$, or $c-d=cd/2$. So $d=2c/(c+2)$. Finally, we can plug this into $s=d^2/4$ to obtain

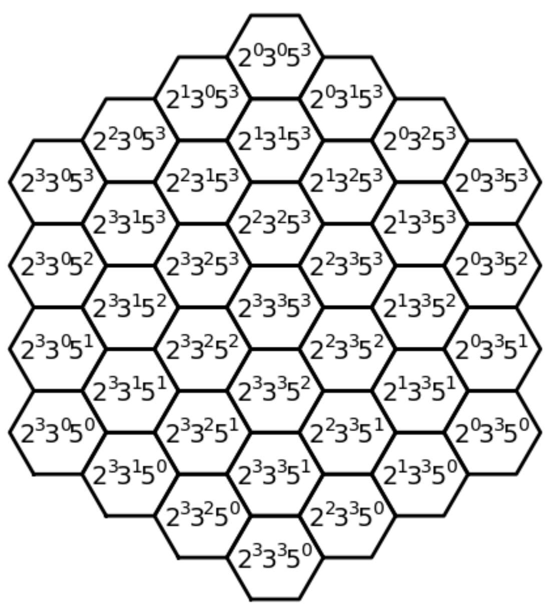
$$\lim_{c\to 0}\frac{s(c)}{r(c)}=\lim_{c\to 0}\frac{\frac{1}{4}\left(\frac{2c}{c+2}\right)^2}{\frac{1}{4}c^2}=\lim_{c\to 0}\frac{4}{(c+2)^2}=1.$$

7. Let S be the collection of positive integers that are divisors of 27,000 but not of 900. We will say that two elements of S are *compatible* if one is a multiple of the other.

Archimedes and Beethoven are playing a game in which they assign colors to elements of S. A number can be colored red if it is a multiple of 2, blue if it is a multiple of 3, and green if it is a multiple of 5. A player colors at least one number on his or her turn, and once a number has been assigned a color, that color may not be changed. If, after a player's turn, there are three numbers, pairwise compatible, with three different colors, then that player loses.

Archimedes plays first. Describe a winning strategy for one of the two players.

Solution: The elements of S are of the form $2^a3^b5^c$, with $0\le a,b,c\le 3$, but the condition that an element of S not be a divisor of 900 necessitates that at least one of a,b,c be 3. The set S can be visualized as follows: mapping $2^a3^b5^c$ to (a,b,c), the divisors of 27000 form a cube. The elements of S in particular form the three faces of the cube lying in the $x=3, y=3$, and $z=3$ planes. If we look at the cube from far out on the line $x=y=z$, we see a hexagon:



Hexagons are chosen for this diagram to highlight the "compatibility" relationship: adjacency implies compatibility. Our strategy below does not require us to prove this.

Archimedes defeats Beethoven by coloring all but one of the elements of S in such a fashion that, no matter what color Beethoven assigns to the remaining number, he creates a set of three mutually compatible numbers with different colors. (There are many ways to do this.)

Specifically, let $R = \{x \in S : 2^3 \mid x \text{ but } 3^3 \nmid x\}$, $B = \{x \in S : 3^3 \mid x \text{ but } 5^3 \nmid x\}$, and $G = \{x \in S : 5^3 \mid x \text{ but } 2^3 \nmid x\}$. These three sets are pairwise disjoint, so Archimedes (legally) colors each element of R red, B blue, and G green. Moreover, in doing so, Archimedes does not lose: if $m \in R$ and $n \in B$, then n is divisible by 3^3 and m is not. In order for m and n to be compatible, one must be a divisor of the other, so we must have $m \mid n$. Similarly, for $n \in B$ and $p \in G$ to be compatible, we must have $n \mid p$, and for $p \in G$ and $m \in R$ to be compatible, we must have $p \mid m$. Certainly at most two of these divisibility relations can hold simultaneously, so Archimedes has not colored foolishly.

The only element of S that is not in $R \cup B \cup G$ is 27000, or $2^3 3^3 5^3$. Archimedes has colored $2^2 3^3 5^3$ green and $2^2 3^3 5^2$ blue; these are compatible with 27000 and with each other, so Beethoven must not color 27000 red. Likewise, $2^3 3^2 5^3$ is red and $2^2 3^2 5^3$ is green, so Beethoven must not color 27000 blue. Also, $2^3 3^2 5^2$ is red and $2^3 3^3 5^2$ is blue, so Beethoven must not color 27000 green. Whichever color Beethoven chooses, he loses.

8. Prove that every (real-valued) solution of the differential equation $y''' = y$ has either no real roots, or infinitely many.

Solution: Consider $e^x, e^{\omega x}$, and $e^{\omega^2 x}$, where $\omega = \dfrac{-1+\sqrt{3}i}{2}$. By inspection, these are all solutions to $y''' = y$. Using an Euler's formula, we rewrite $e^{\omega x}$ as

$$e^{\omega x} = e^{-x/2} \cdot e^{i(\sqrt{3}/2)x} = e^{-x/2}\left(\cos\frac{\sqrt{3}}{2}x + i\sin\frac{\sqrt{3}}{2}x\right).$$

Similarly, (since $\omega^2 = \bar{\omega}$), $e^{\omega^2 x} = e^{-x/2}\left(\cos\frac{\sqrt{3}}{2}x - i\sin\frac{\sqrt{3}}{2}x\right)$. The DE $y''' = y$ is homogeneous, so taking linear combinations of these, we have the real-valued functions $e^{-x/2}\cos\left(\dfrac{\sqrt{3}}{2}x\right)$ and $e^{-x/2}\sin\left(\dfrac{\sqrt{3}}{2}x\right)$, which are also solutions. Together with e^x, these form a linearly independent set, and there are three of them, so they form a basis over $\mathbb{R}$ for the space of all solutions of $y''' = y$. That is, every real-valued solution to $y''' = y$ can be written in the form

$$f(x) = Ae^x + Be^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + Ce^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right).$$

If $B = C = 0$, then $f(x)$ is of the form Ae^x, which has no real roots.
Otherwise, $B^2 + C^2 > 0$. Taking $\alpha = \operatorname{sgn}(C)\arccos\left(B/\sqrt{B^2+C^2}\right)$, we rewrite f as

$$f(x) = Ae^x + \sqrt{B^2+C^2}\, e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x - \alpha\right),$$

which is zero whenever

$$\cos\left(\frac{\sqrt{3}}{2}x - \alpha\right) = \frac{-A}{\sqrt{B^2+C^2}}e^{3x/2}.$$

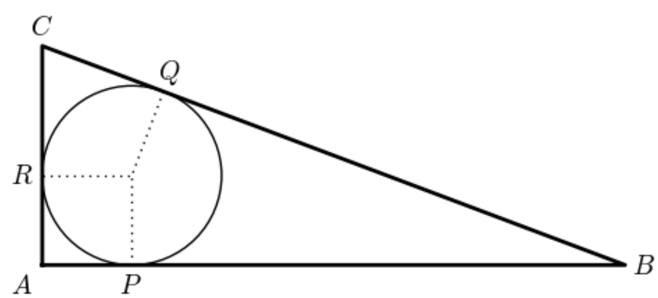
If $A = 0$, this has a solution whenever $\frac{\sqrt{3}}{2}x - \alpha$ is an odd multiple of $\pi/2$. Otherwise, consider that $\lim_{x\to-\infty} e^{3x/2} = 0$, so eventually $e^{3x/2} < \sqrt{B^2+C^2}/|A|$. Let N be a (negative) integer such that $x = N\pi$ satisfies this inequality. Then for any $n < N$, the cosine takes on all values between -1 and 1 on the interval $-n\pi \leq \frac{\sqrt{3}}{2}x - \alpha \leq (-n+1)\pi$, while $\frac{-A}{\sqrt{B^2+C^2}}e^{3x/2}$ is strictly between -1 and 1. So $f(x)$ has a root on each such interval. Therefore, any such f has infinitely many roots.

9. A right triangle T has integer side lengths, none greater than 400, with no common factor. If the average distance from a point on the interior of T to its boundary is 6, what are the lengths of the sides of T?

Solution: Denote by $z(x, y)$ the distance from a point (x, y) on the interior of T to its boundary. Then the average value of z is given by

$$\overline{z} = \frac{1}{\text{Area}(T)} \iint_T z(x, y)\, dA.$$

However, since the distance from an edge increases linearly as we move away from that edge, the volume described by the double integral is that of a tetrahedron with base T and height h, where h is the maximum distance from a point in T to the boundary. Therefore, $\overline{z} = \frac{1}{\text{Area}(T)} \cdot \frac{1}{3}\,\text{Area}(T) \cdot h = \frac{h}{3}$. Since we are given $\overline{z} = 6$, it follows $h = 18$.
Moreover, h is the radius of the largest circle inscribed in T, the center being the point where the distances to all three edges are equal. Let A, B, and C denote the vertices of the triangle, and P, Q, and R denote the points of tangency of the incircle, as shown in the figure below. Since $|BP| = |BQ|$ and $|CR| = |CQ|$, we can conclude that $h = |AP| = |AR| = \frac{1}{2}(|AB| + |AC| - |BC|)$.



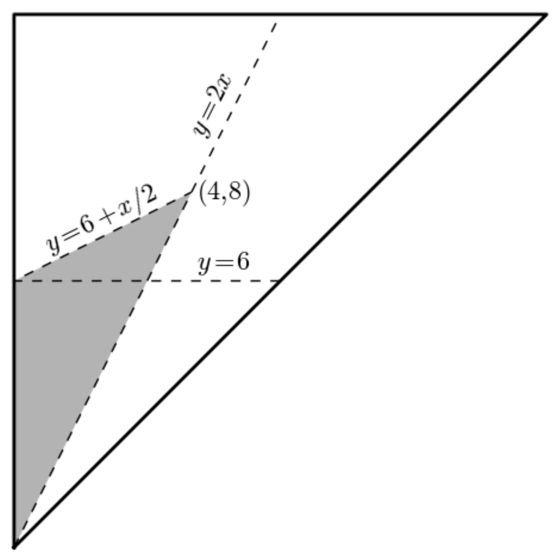
Using the well-known parametrization of Pythagorean triples, we know there are integers m and n such that $|AB| = m^2 - n^2$, $|AC| = 2mn$, and $|BC| = m^2 + n^2$. Then $\frac{1}{2}(|AB| + |AC| - |BC|)$ simplifies to $n(m-n)$, which is, as above, 18. If m and n are the same parity, then $m^2 - n^2, 2mn$, and $m^2 + n^2$ have a common factor of 2, so $m - n$ must be odd. If $m - n = 3$, then n must be 6, which makes all three sides multiples of 3. Thus the only two possible factorizations of 18 that apply are $2 \cdot 9$ and $18 \cdot 1$. In the latter case, $m = 19$ and $n = 18$, which makes the hypotenuse $19^2 + 18^2 = 685 > 400$. So we want the other case, which yields $m = 11$ and $n = 2$. Therefore, the lengths of the sides of T are $m^2 - n^2 = 117$, $2mn = 44$, and $m^2 + n^2 = 125$.

10. Legend has it that the first Konhauser Problemfest was carried through the desert by four camels named Mac, Carl, Ole, and Tom. Their journey was long and difficult, but they were aided by the presence of three oases along their route, so Mac carried the problems until the first oasis, Carl from the first to the second, Ole from the second to the third, and Tom the rest of the way. Exact details have been buried by the sands of time, so we no longer know how the oases were positioned. Assuming an equal likelihood of an oasis at any point along the route, what is the probability that Carl carried the load further than Mac, and Tom carried the load further than Carl, but not as far as the other three camels combined?

Solution: Without loss of generality, let the total distance travelled be 12. Denote the positions of the oases by x, y, z where $0 < x < y < z < 12$; then Mac carried the problems for a distance x, Carl for $y - x$, Ole for $z - y$, and Tom for $12 - z$.

The sample space $S = \{(x, y, z) \in \mathbb{R}^3 : 0 < x < y < z < 12\}$ is a tetrahedron whose base (in the $z = 12$ plane) is an isosceles right triangle with area $\frac{1}{2}12^2$ and height 12, so its volume is $\frac{1}{6}12^3 = 288$. The conditions imposed in the problem give rise to the inequalities $12 - z > y - x > x$, and $12 - z < (z - y) + (y - x) + x$ (or, equivalently, $z > 6$). If we write $g(x, y) = 12 - y + x$, then for $(x, y, z) \in S$ we can say that Tom carried the load further than Carl, but not as far as the other three camels combined if and only if $6 < z < g(x, y)$. Also, Carl carried the load further than Mac if and only if $y > 2x$. Accordingly, we define $A = \{(x, y, z) \in S : 6 < z < g(x, y), y > 2x\}$; the desired probability is $\dfrac{\text{vol}(A)}{\text{vol}(S)}$.

The projection of A onto the xy-plane lies in the triangle bounded by the lines $y = 2x$, $x = 0$, $y = 12$. However, if $y > 6 + x/2$, then $y > 12 - y + x$, but we must also have $z \geq y$; together these contradict Tom's carrying the load further than Carl. The projection of A is thus as shown.



Now we can calculate vol(A) as

$$\begin{aligned}
\text{vol}(A) &= \int_0^6 \int_0^{y/2} (g(x,y)-6)\ dx\,dy + \int_6^8 \int_{2y-12}^{y/2} (g(x,y)-y)\ dx\,dy \\
&= \int_0^6 \int_0^{y/2} (6-y+x)\ dx\,dy + \int_6^8 \int_{2y-12}^{y/2} (12-2y+x)\ dx\,dy \\
&= \int_0^6 \left[(6-y)\left(\frac{y}{2}\right) + \frac{1}{2}\left(\frac{y}{2}\right)^2\right] dy + \int_6^8 \left[(12-2y)\left(\frac{y}{2} - (2y-12)\right) + \frac{1}{2}\left(\left(\frac{y}{2}\right)^2 - (2y-12)^2\right.\right. \\
&= \int_0^6 \left(3y - \frac{3}{8}y^2\right) dy + \int_6^8 \left(72 - 18y + \frac{9}{8}y^2\right) dy \\
&= \left(\frac{3}{2}y^2 - \frac{1}{8}y^3\right)\Bigg|_0^6 + \left(72y - 9y^2 + \frac{3}{8}y^3\right)\Bigg|_6^8 \\
&= 27 + 3 = 30.
\end{aligned}$$

The probability is therefore $\frac{30}{288}$, or $\frac{5}{48}$.

Alternate Solution: With x, y, z as above, let $A = x, B = y - x, C = z - y, D = 12 - z$.

Let S be the event that $D > B > A$. Let T be the event that $D < (A + B + C)$. We want $\Pr(S \cap T)$. By the Inclusion–Exclusion Principle, this is $\Pr(S) + \Pr(T) - \Pr(S \cup T)$.

First, we consider $\Pr(S)$. All 24 possible orderings of these lengths are equally likely. There are four that meet the needed conditions: $DCBA$, $DBCA$, $DBAC$, and $CDBA$. Therefore, $\Pr(S) = 4/24$.

Now, $\Pr(T)$: this is just the probability that $D < 6$, which is the probability that at least one of the randomly-situated oases is more than halfway along the journey. The probability that this is *not* the case is $(1/2)^3$; therefore, $\Pr(T) = 1 - (1/2)^3 = 7/8$.

Finally, $\Pr(S \cup T)$. Suppose T doesn't occur. This means $D > 6$ which, in particular, guarantees that $D > B$ and $D > A$. So if T doesn't occur, then S occurs if and only if $B > A$. This means $\Pr(S \cup T) = \Pr(T) + \Pr(B > A \cap T^c)$. As there are no other conditions distinguishing Mac and Carl, $\Pr(B > A \cap T^c) = \Pr(A > B \cap T^c)$, so $\Pr(S \cup T) = 7/8 + 1/2(1/8) = 15/16$.

So, by inclusion/exclusion, $\Pr(S \cap T) = 1/6 + 7/8 - 15/16 = 5/48$.