Twenty-Fourth Annual Konhauser Problemfest

Macalester College **Saturday, Feb. 27, 2016**

Problems by Stan Wagon (Macalester College, retired)

Correct answers alone are not worth many points. Full details of why the answer is correct are required, except in #8. For #8, the answer is sufficient and can be entered on the hand-in sheet with a blank table. Calculators are allowed.

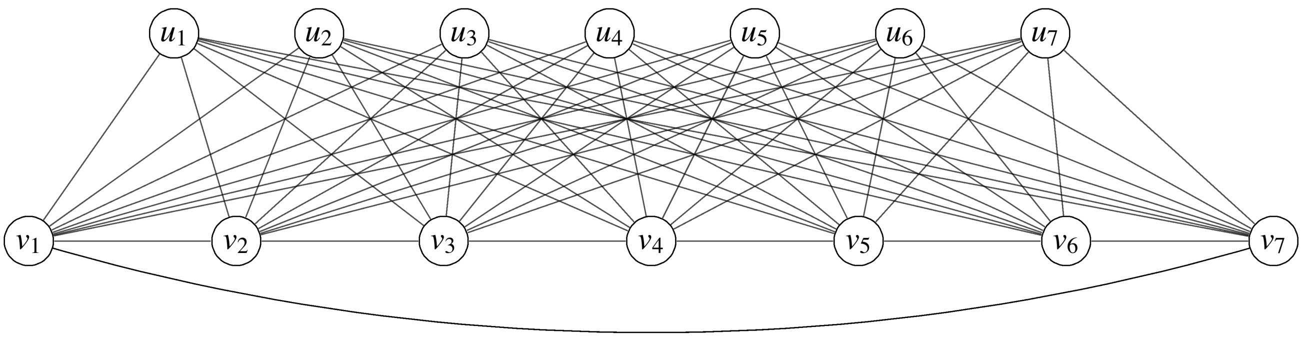
1. Footworms A footworm grows at a constant rate of one foot per day and stops growing when it reaches one foot. A full-grown worm can be cut into two worms of lengths x and $1 - x$, each of which then grows at the constant rate until full-grown. Worms that are not full-grown cannot be cut. Show how to produce, in one day, six full-grown worms, where you start with one full-grown worm.

2. Football is a Zero-Sum-Square Game The MIAC football standings for 2015 are in the table, where W denotes wins and L denotes losses. Football is a zero-sum game (there are no ties), so it must be that $\Sigma W = \Sigma L$ (36 in the table). But we see also that $\Sigma(W^2) = \Sigma(L^2)$ (200 in the table). Prove that in any tournament of n football teams, where each team plays each other team once, this relation holds: $\Sigma(W^2) = \Sigma(L^2)$.

	W	L	W^2	L^2
Team 1	8	0	64	0
Team 2	7	1	49	1
Team 3	5	3	25	9
Team 4	5	3	25	9
Team 5	4	4	16	16
Team 6	4	4	16	16
Team 7	2	6	4	36
Team 8	1	7	1	49
Team 9	0	8	0	64
sums	36	36	200	200

3. A Thorny Crown Define the crown graph $C_{m,n}$ to consist of an n-cycle with vertices v_i, and m additional vertices u_j that are initially isolated. Then edges are added between each u_i and v_j; the diagram shows $C_{3,7}$. Show how to color the edges of $C_{7,7}$ using the smallest number of colors so that edges with a common vertex get different colors. Details on why the coloring works or a proof that it is smallest are not required. Partial credit will be given for correct colorings that are not minimal.

The crown graph $C_{7,7}$

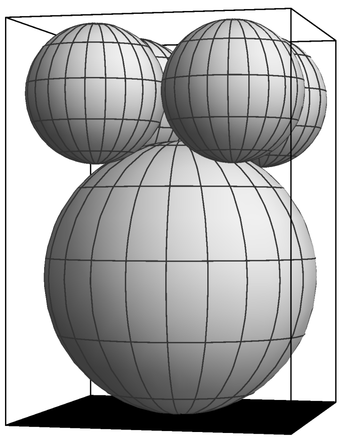


Each blank square in the table below corresponds to an edge of $C_{7,7}$. The bottom row has the cycle edges: $v_1 \bullet\!\!-\!\!\bullet v_2$, $v_2 \bullet\!\!-\!\!\bullet v_3$, and so on, up to $v_7 \bullet\!\!-\!\!\bullet v_1$at bottom right. Fill in the squares (use the attached hand-in sheet) with your colors, using 1, 2, 3,... for the colors.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
u_1							
u_2							
u_3							
u_4							
u_5							
u_6							
u_7							

cycle:							
	$v_1 \bullet\!\!-\!\!\bullet v_2$			$v_4 \bullet\!\!-\!\!\bullet v_5$			$v_7 \bullet\!\!-\!\!\bullet v_1$

4. Fruit Salad A cantaloupe melon is a sphere of radius 4 inches and a grapefruit is a sphere of radius 2 inches. You want to pack one cantaloupe and four grapefruit into a box whose base is a square that is 8 inches by 8 inches. First you put in the cantaloupe, and then you put the four grapefruit on top of it in one layer. Find the height of the shortest box that will contain the fruit and allow a top to be placed on it.



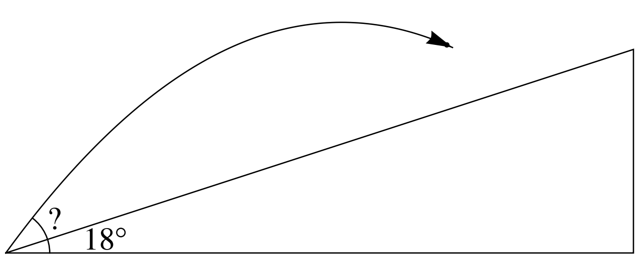
5. How Much is a Penny Worth? Alice tosses 99 fair coins and Bob tosses 100. What is the probability that Bob gets more (meaning: strictly more) heads than Alice? Express your answer as a fraction $\frac{m}{n}$.

6. An Arithmetic Problem An *arithmetic sequence* is a sequence $(x_n)_{n=1}^{\infty}$ with constant differences: for every m and n, the differences $x_{m+1} - x_m$ and $x_{n+1} - x_n$ are equal. Let $(a_n)_{n=1}^{\infty}$ be a strictly increasing arithmetic sequence of positive real numbers; for each n let $b_n = \frac{a_1+a_2+\cdots+a_n}{a_{n+1}+a_{n+2}+\cdots+a_{2n}}$.

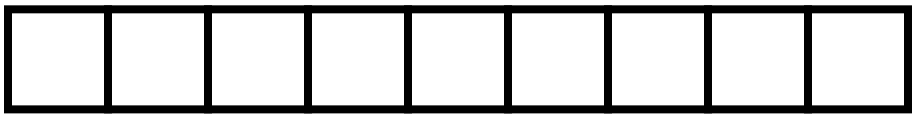
(a) Find $\lim_{n\to\infty} b_n$.

(b) For which initial sequences $\{a_n\}$ is the sequence $\{b_n\}$ a constant sequence?

7. A T-Shirt Gun The stands at a stadium slope up at an 18° angle. You are firing a T-shirt cannon from the bottom of the stands into the stands. At what angle of elevation above the ground should you fire the cannon to get the shirt to land as far up the stands as possible? Ignore air resistance. The answer is $n°$ where n is an integer, so your answer should specify the value of n.



8. Help Alice and Bob play a game on a row of 9 squares, with Alice moving first. On **her** move, Alice plays an S into an empty square; on his move, Bob plays an O into an empty square. The person who first completes SOS in consecutive squares wins. What first move guarantees that Alice will win?



9. Battleship on the Horizon A battleship of length 4 is moving along the real line at constant speed. At time 0, its center is at point X, and it then moves with constant speed V. Both X and V are unknown integers (i.e., in $\{0, \pm 1, \pm 2, \ldots\}$), where negative speed means leftward motion. At every second you can shoot at some number on the real line; if you strike the ship the process ends successfully. Find a strategy that guarantees success in finite time. Assume that your projectile takes zero time to arrive at the aimed-at point.

10. A Choice Problem Let $A = \{1, 2, 3, 4, 5\}$ and let P be the set of all nonempty subsets of A. A function f: $P \to A$ is a *choice function* if:

(1) for every $B \in P$, $f(B) \in B$; and

(2) for every $B, C \in P$, either $f(B \cup C) = f(B)$ or $f(B \cup C) = f(C)$

How many choice functions are there?

Hand-in Sheet for #3

Team Code ________________________

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
u_1							
u_2							
u_3							
u_4							
u_5							
u_6							
u_7							

cycle:							

$v_1 \bullet\!\!-\!\!\bullet v_2$ $v_4 \bullet\!\!-\!\!\bullet v_5$ $v_7 \bullet\!\!-\!\!\bullet v_1$

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
u_1							
u_2							
u_3							
u_4							
u_5							
u_6							
u_7							

cycle:							

$v_1 \bullet\!\!-\!\!\bullet v_2$ $v_4 \bullet\!\!-\!\!\bullet v_5$ $v_7 \bullet\!\!-\!\!\bullet v_1$

24th Annual Konhauser Problemfest Problems and Solutions

Macalester College, Feb. 27, 2016 Problems by Stan Wagon (Macalester College, retired)

1. Footworms A footworm grows at a constant rate of one foot per day and stops growing when it reaches one foot. A full-grown worm can be cut into two worms of lengths x and $1-x$, each of which then grows at the constant rate until full-grown. Worms that are not full-grown cannot be cut. Show how to produce, in one day, six full-grown worms, where you start with one full-grown worm.

Solution Let the unit of time be days. Use superscripts $^-$ and $^+$ to indicate the situation just before and just after the current cutting move. Then the chart below shows how to get six worms in $\frac{31}{32}$ of a day. In short, a cut occurs whenever there is a foot-long worm. At the nth cut the footworm is split into lengths 2^{n-6} and $1-2^{n-6}$. This gets six worms of length $\frac{1}{2}$ at time $\frac{15}{32}$, and they grow to become footworms before the end of the day.

Time 0^-: 1	Time 0^+: $\frac{1}{32}\ \frac{31}{32}$
Time $\left(\frac{1}{32}\right)^-$: $\frac{1}{16}\ 1$	Time $\left(\frac{1}{32}\right)^+$: $\frac{1}{16}\ \frac{1}{16}\ \frac{15}{16}$
Time $\left(\frac{3}{32}\right)^-$: $\frac{1}{8}\ \frac{1}{8}\ 1$	Time $\left(\frac{3}{32}\right)^+$: $\frac{1}{8}\ \frac{1}{8}\ \frac{1}{8}\ \frac{7}{8}$
Time $\left(\frac{7}{32}\right)^-$: $\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}\ 1$	Time $\left(\frac{7}{32}\right)^+$: $\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}\ \frac{3}{4}$
Time $\left(\frac{15}{32}\right)^-$: $\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ 1$	Time $\left(\frac{15}{32}\right)^+$: $\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}$
Time $\frac{31}{32}$: $1\ 1\ 1\ 1\ 1\ 1$	

This method shows that one can start with 2^{-n} instead of 2^{-5}, and so get a googol, or more, worms in one day.

There are many many other ways to do it. In inches and hours the growth rate is $\frac{1}{2}$ inch per hour or $\frac{1}{10}$ inch in every $\frac{1}{5}$ hour, so the following yields the 6 worms in exactly 24 hours.

Time 0^-: 12	Time 0^+: 11.9 0.1
Time $\left(\frac{1}{5}\right)^-$: 12 0.2	Time $\left(\frac{1}{5}\right)^+$: 11.9 0.1 0.2
Time $\left(\frac{2}{5}\right)^-$: 12 0.2 0.3	Time $\left(\frac{2}{5}\right)^+$: 11.8 0.2 0.2 0.3
Time $\left(\frac{4}{5}\right)^-$: 12 0.4 0.4 0.5	Time $\left(\frac{4}{5}\right)^+$: 11.6 0.4 0.4 0.4 0.5
Time $\left(\frac{8}{5}\right)^-$: 12 0.8 0.8 0.8 1	Time $\left(\frac{8}{5}\right)^+$: 6 6 0.8 0.8 0.8 1
Time 24: 17.2 17.2 12 12 12 12.2	

which is really 12, 12, 12, 12, 12, 12 since a footworm cannot exceed a foot.

Or: Work in hours and feet: growth rate is $\frac{1}{24}$ of a foot per hour. Use times 0, 1, 2, 4, 8, 16, 24 hours as follows:

Time 0^-: 1	Time 0^+: $\frac{23}{24}\ \frac{1}{24}$
Time 1^-: $1\ \frac{2}{24}$	Time 1^+: $\frac{23}{24}\ \frac{1}{24}\ \frac{2}{24}$
Time 2^-: $1\ \frac{2}{24}\ \frac{3}{24}$	Time 2^+: $\frac{22}{24}\ \frac{2}{24}\ \frac{2}{24}\ \frac{3}{24}$
Time 4^-: $1\ \frac{4}{24}\ \frac{4}{24}\ \frac{5}{24}$	Time 4^+: $\frac{20}{24}\ \frac{4}{24}\ \frac{4}{24}\ \frac{4}{24}\ \frac{5}{24}$
Time 8^-: $1\ \frac{8}{24}\ \frac{8}{24}\ \frac{8}{24}\ \frac{9}{24}$	Time 8^+: $\frac{16}{24}\ \frac{8}{24}\ \frac{8}{24}\ \frac{8}{24}\ \frac{8}{24}\ \frac{9}{24}$
Time 24: $1\ 1\ 1\ 1\ 1\ 1$	

2. Football is a Zero-Sum-Square Game The MIAC football standings for 2015 are in the table, where W denotes wins and L denotes losses. Football is a zero-sum game (there are no ties), so it must be that $\Sigma W = \Sigma L$ (36). But we see also that $\Sigma(W^2) = \Sigma(L^2)$ (200). Prove that in any tournament of n football teams, where each team plays each other team once, this relation holds: $\sum(W^2) = \sum(L^2)$.

	W	L	W^2	L^2
Team 1	8	0	64	0
Team 2	7	1	49	1
Team 3	5	3	25	9
Team 4	5	3	25	9
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Team 6	4	4	16	16
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sums	36	36	200	200

Solution. Let there be n teams; let W_i be the number of wins of the ith team, and L_i the losses. Then $\sum W_i - \sum L_i = 0$, and also $W_i + L_i = n - 1$ for each team. Now

$$\begin{aligned}\sum W_i^2 - \sum L_i^2 &= \sum(W_i^2 - L_i^2) = \sum(W_i + L_i)(W_i - L_i) = \sum(n-1)(W_i - L_i)\\ &= (n-1)\sum(W_i - L_i) = (n-1)\left(\sum W_i - \sum L_i\right) = 0\end{aligned}$$

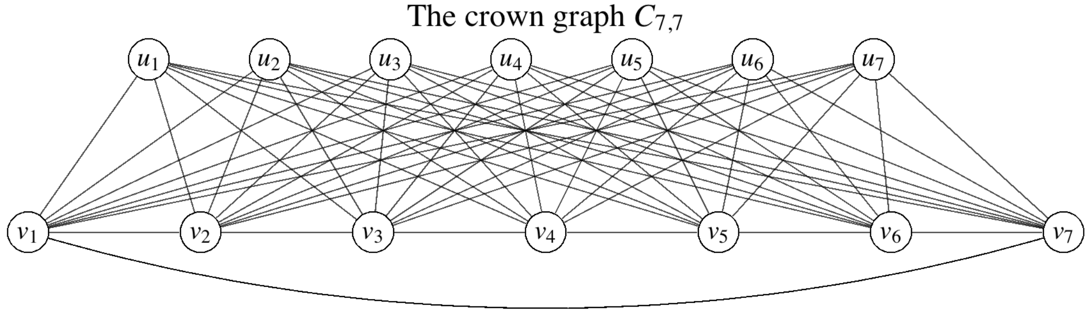
An alternate approach is to show that the result is true for the special tournament where the best record is $n - 1$ wins and no losses, the next best is $n - 2$ wins and 1 loss, and so on down to 0 wins and $n - 1$ losses. Then show that the quantity $\sum W_i^2 - \sum L_i^2$ does not change when the results of a single game are flipped. For this, consider a result-switch of the game between teams i and j (assume i lost). That turns the vector (W_i, L_i, W_j, L_j) into $(W_i + 1, L_i - 1, W_j - 1, L_j + 1)$. Square the two vectors and compare. The impact of the first on $\sum W_i^2 - \sum L_i^2$ is $W_i^2 - L_i^2 + W_j^2 - L_j^2$, which we can call X. For the second we have

$$\begin{aligned}&(W_i + 1)^2 - (L_i - 1)^2 + (W_j - 1)^2 - (L_j + 1)^2 =\\ &X + 2W_i + 2L_i - 2W_j + 2L_j = X + 2(W_i + L_i) - 2(W_j + L_j) = X + 2(n-1) - 2(n-1) = X\end{aligned}$$

So there is overall no change. Then observe that any season-standings table can be obtained by a sequence of such two-team win-loss switches.

3. A Thorny Crown Define the crown graph $C_{m,n}$ to consist of an n-cycle with vertices v_i, and m additional vertices u_j that are initially isolated. Then edges are added between each u_i and v_j; the diagram shows $C_{3,7}$. Show how to color the edges of $C_{7,7}$ using the smallest number of colors so that edges with a common vertex get different colors. Details on why the coloring works or a proof that it is smallest are not required. Partial credit will be given for correct colorings that are not minimal.

The crown graph $C_{7,7}$



Each blank square in the table below corresponds to an edge of $C_{7,7}$. The bottom row has the cycle edges: $v_1 \bullet\!\!-\!\!\bullet v_2$, $v_2 \bullet\!\!-\!\!\bullet v_3$, and so on, up to $v_7 \bullet\!\!-\!\!\bullet v_1$at bottom right. Fill in the squares (use the attached hand-in sheet) with your colors, using 1, 2, 3,… for the colors.

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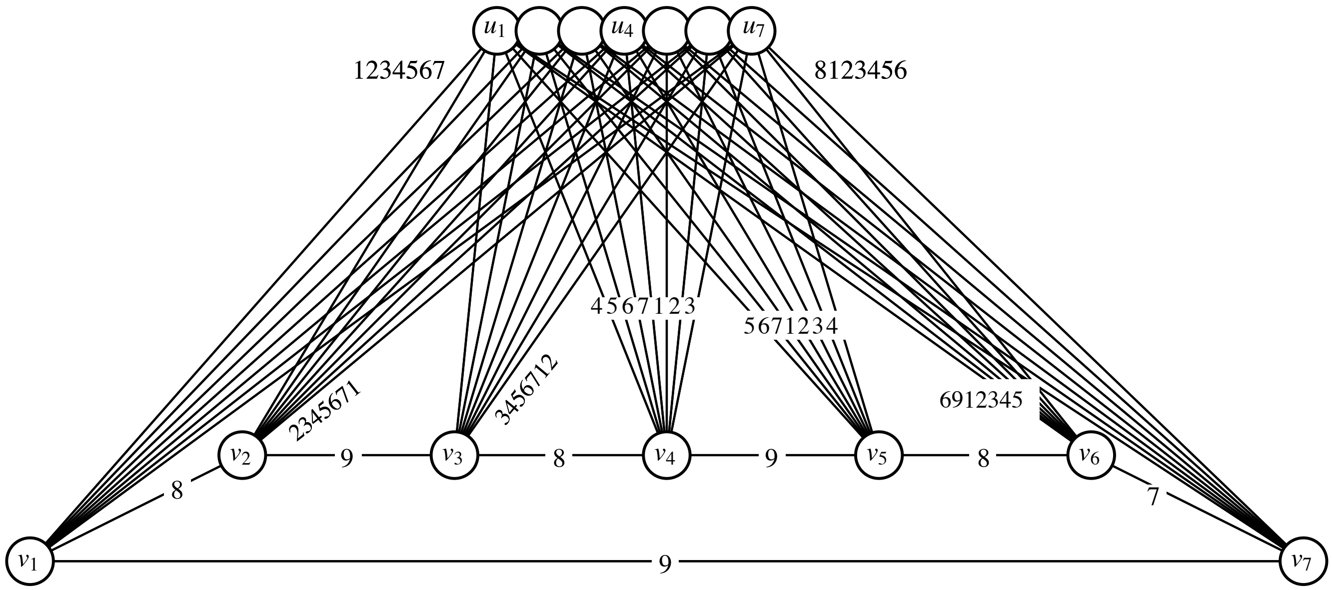
Solution. Each cycle vertex v_i is connected to 2 other cycle vertices and all 7 of the u-vertices. Therefore it has degree $2+7=9$, and so a coloring with fewer than 9 colors is impossible (a proof of this was not required, but the problem did ask for the smallest number of colors). One way to do it with 9 colors is as shown. The 7-cycle requires 3 colors (because 7 is odd), so one can do that first (using 8989879), and then work cyclically through the rest, making adjustments as needed.

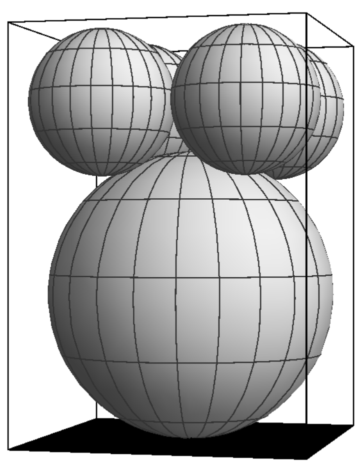
	v_1	v_2	v_3	v_4	v_5	v_6	v_7
u_1	1	2	3	4	5	6	8
u_2	2	3	4	5	6	9	1
u_3	3	4	5	6	7	1	2
u_4	4	5	6	7	1	2	3
u_5	5	6	7	1	2	3	4
u_6	6	7	1	2	3	4	5
u_7	7	1	2	3	4	5	6

cycle:	8	9	8	9	8	7	9
	$v_1 \bullet\!\!-\!\!\bullet v_2$			$v_4 \bullet\!\!-\!\!\bullet v_5$			$v_7 \bullet\!\!-\!\!\bullet v_1$

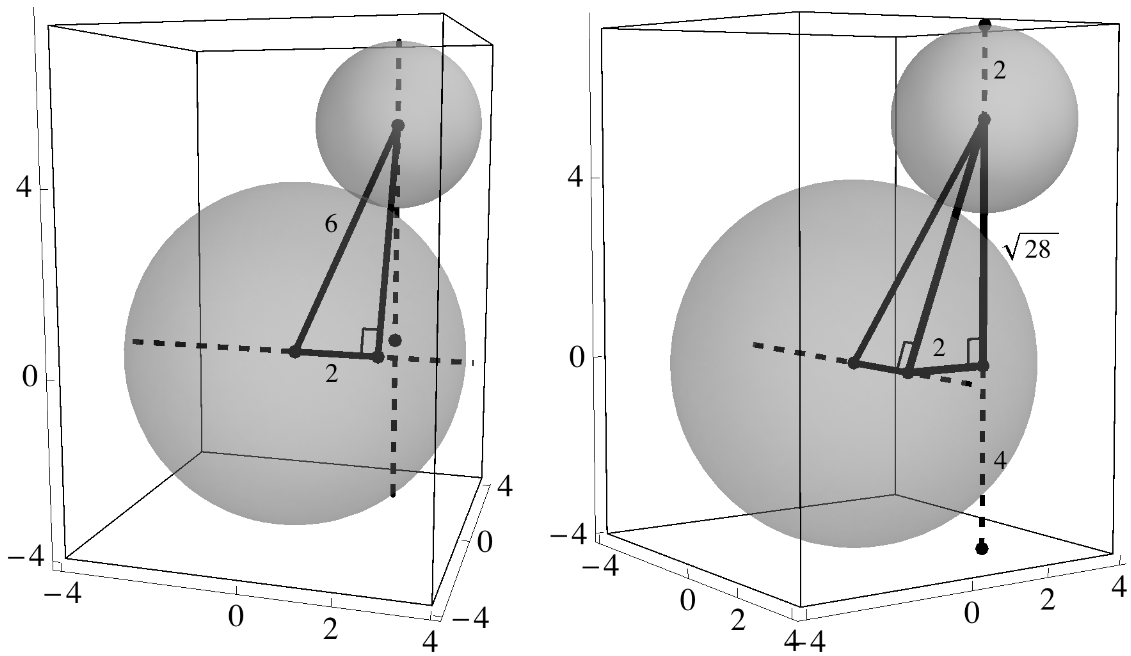
A direct view of the edge coloring is this:



4. Fruit Salad A cantaloupe melon is a sphere of radius 4 inches and a grapefruit is a sphere of radius 2 inches. You want to pack one cantaloupe and four grapefruit into a box whose base is a square that is 8 inches by 8 inches. First you put in the cantaloupe, and then you put the four grapefruit on top of it in one layer. Find the height of the shortest box that will contain the fruit and allow a top to be placed on it.



Solution. The height should be $6 + 2\sqrt{7}$. Assume the radius-4 sphere is centered at (0, 0, 0). Then the radius-2 grapefruit have centers $(\pm 2, \pm 2, h)$; consider the one at $(2, 2, h)$. The line from this point to the origin is the hypotenuse of a right triangle, with right angle at (2, 0, 0). Its length is the sum of the radii: $4 + 2 = 6$. Therefore the nonhorizontal leg has length $\sqrt{6^2 - 2^2} = \sqrt{32}$. This is the hypotenuse of another right triangle in the plane $x = 2$ with length-2 horizontal leg. So the vertical leg length is $\sqrt{32 - 4} = \sqrt{28}$. Adding in the two radii gives the box height as $\sqrt{28} + 4 + 2 = 6 + \sqrt{28} = 6 + 2\sqrt{7} \sim 11.3$.



5. How Much is a Penny Worth? Alice tosses 99 fair coins and Bob tosses 100. What is the probability that Bob gets more (meaning: strictly more) heads than Alice? Express your answer as a fraction $\frac{m}{n}$.

Solution. The answer is exactly $\frac{1}{2}$. An informal way of seeing this is to consider the case where Bob has one coin and Alice has none; the answer is clearly $\frac{1}{2}$. The additional 99 tosses by the two of them do not affect the result. The result is true for any consecutive integers in place of 99 and 100. More formally one can argue as follows.

Let X be the probability that, after 99 throws by each, Alice has more heads. By symmetry, X is also the probability that after 99 throws by each, Bob has more heads. Therefore $1-2X$ is the probability of a tie at this point. The probability (P) that Bob wins the full game is:

P(Bob leads after the first 99) + P(a tie after first 99) $\cdot$ P(Bob's last throw is a head)

This is $X+\frac{1}{2}(1-2X)=X+\frac{1}{2}-X=\frac{1}{2}$. □

6. An Arithmetic Problem An *arithmetic sequence* is a sequence $(x_n)_{n=1}^{\infty}$ with constant differences: for every m and n, the differences $x_{m+1}-x_m$ and $x_{n+1}-x_n$ are equal. Let $(a_n)_{n=1}^{\infty}$ be a strictly increasing arithmetic sequence of positive real numbers; for each n let $b_n=\frac{a_1+a_2+\cdots+a_n}{a_{n+1}+a_{n+2}+\cdots+a_{2n}}$.

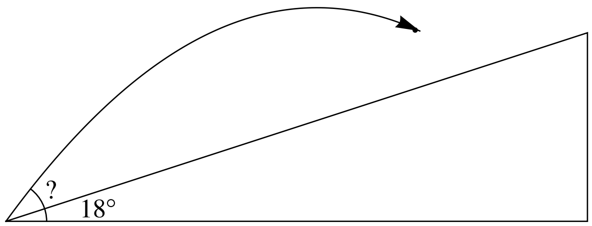
(a) Find $\lim_{n\to\infty} b_n$.

(b) For which initial sequences $\{a_n\}$ is the sequence $\{b_n\}$ a constant sequence?

Solution. (a) $\lim_{n\to\infty} b_n=\frac{1}{3}$. Because $a_i=a+i\,d$ and $\sum_{i=0}^{n-1} i=\frac{1}{2}n(n-1)$, the numerator of b_n is $\sum_{i=0}^{n-1}(a+i\,d)=n\,a+\frac{1}{2}d\,n(n-1)=\frac{1}{2}n(2a-d+d\,n)$; similarly the denominator is $\frac{1}{2}n(2a+3d\,n-d)$. Therefore $b_n=\frac{2a-d+d\,n}{2a-d+3d\,n}=\frac{\frac{2a-d}{n}+d}{\frac{2a-d}{n}+3d}\to\frac{d}{3d}=\frac{1}{3}$ as $n\to\infty$.

(b) If b_n is constant, then the constant must be $\frac{1}{3}$. So $b_1 = \frac{1}{3}$. But $b_1 = \frac{a_1}{a_2}$. So $a_2 = 3a_1$ and the difference of the first two terms is $2\,a_1$. Since the difference never changes, the original sequence looks like $a, 3\,a, 5\,a, 7\,a, \ldots$. Another way of saying this is: $a = \frac{d}{2}$ (or $d = 2\,a$) where a is the first term and d the common difference.

7. A T-Shirt Gun The stands at a stadium slope up at an 18° angle. You are firing a T-shirt cannon from the bottom of the stands into the stands. At what angle of elevation above the ground should you fire the cannon to get the shirt to land as far up the stands as possible? Ignore air resistance. The answer is $n°$ where n is an integer, so your answer should specify the value of n.



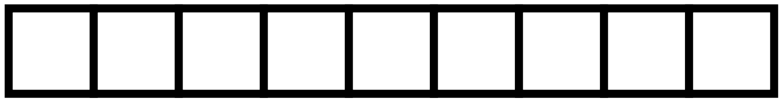
Solution. Let v be the (unknown) initial velocity of the shirt, g the acceleration of gravity, and θ the angle of elevation of the gun; let $\tau = \tan 18°$. The position of the shirt at time t is $x = v\,t\cos\theta$, $y = v\,t\sin\theta - \frac{1}{2}\,g\,t^2$. The shirt strikes the stands (we may assume here that $\theta \geq 18\,°$) when $\tau = \frac{y}{x} = \frac{v\,\sin\theta - \frac{1}{2}\,g\,t}{v\cos\theta}$, so $t = \frac{1}{g}2\,v(\sin\theta - \tau\cos\theta)$. The x-coordinate when it lands is therefore

$$x = v\,t\cos\theta = \frac{1}{g}2\,v^2(\sin\theta - (\cos\theta)\,\tau)\cos\theta$$

We seek θ to maximize $\cos\theta\sin\theta - (\cos^2\theta)\,\tau$. The derivative is $\cos(2\,\theta) + \tau\sin(2\,\theta)$, which vanishes only for θ_0 satisfying $\tan(2\,\theta_0) = -\frac{1}{\tau}$. It is physically clear that a maximum exists, so it must occur for θ_0. Note that $\tan(108°) = -\cot 18° = -\frac{1}{\tau}$, which means that $2\,\theta_0 = 108°$ and $\theta_0 = 54°$.

More generally, this argument shows that if the angle of the stands is $A°$, then the optimum gun angle is $45° + \frac{1}{2}A°$. Note that this general answer is the same as saying: Take the average of $A°$ and $90°$; or: Take half of angle $90\,° - A°$ and add that to $A°$. Of course, when there is no incline this is 45°. Is there a simpler geometric of physical argument, avoiding calculus, that the answer is always $45° + \frac{1}{2}A°$?

8. Help Alice and Bob play a game on a row of 9 squares, with Alice moving first. On her move, Alice plays an S into an empty square; on his move, Bob plays an O into an empty square. The person who first completes SOS in consecutive squares wins. What first move guarantees that Alice will win?



Solution. Alice's basic strategy is to form a "pit": a 4-space configuration of the form $S--S$. If Bob can be forced to play into the pit, then Alice wins on the next move. So Alice starts with an S in the fourth square from the left. If Bob moves left of Alice's first move, she moves into square 7, forming a pit with 4 and 7 that Bob will eventually fall into. If Bob moves to her right; then Alice plays an S in the leftmost square. That leaves $S--S$ _ _ _ _ _ where one of the low dashes is an O. Now if Bob plays on the right half, Alice marks the leftmost open square on the right side; this guarantees that Bob won't win. At some point Bob must move into the pit on the left, and Alice then wins.

Extension notes: In the general problem Alice wins on any board of size n where n is odd and $n \geq 7$. Bob wins when n is even and $n \geq 16$. The other cases end in draws.

9. Battleship on the Horizon A battleship of length 4 is moving along the real line at constant speed. At time 0, its center is at point X, and it then moves with constant speed V. Both X and V are unknown integers (i.e., in $\{0, \pm 1, \pm 2, \ldots\}$), where negative speed means leftward motion. At every second you can shoot at some number on the real line; if you strike the ship the process ends successfully. Find a strategy that guarantees success in finite time. Assume that your projectile takes zero time to arrive at the aimed-at point.

Solution. The set of integers is countable (enumerable); also the set of pairs of integers is countable. Enumerate all possible initial states as (X_n, V_n). Now, at integer time n seconds, shoot at $X_n + n V_n$, which will strike the center point of a ship starting at X_n with velocity V_n. Thus every possible initial state will eventually lead to success.

For a more explicit method, use the positive integer $2^X 3^V$ to encode the pair (X, V) when both are nonnegative, use $2^{-X} 3^{-V} 5$ if both are negative, use $2^{-X} 3^V 5^2$ if X is negative and V is not, and use $2^X 3^{-V} 5^3$ if V is negative and X is not. Then, at any positive integer time n, look at the factorization of n; if n does not have one of the four forms above, shoot anywhere. But if n does encode X and V as described above, shoot at $X + n V$, the location of the center of a ship starting at X with velocity V.

10. A Choice Problem Let $A = \{1, 2, 3, 4, 5\}$ and let P be the set of all nonempty subsets of A. A function $f: P \to A$ is a *choice function* if:

(1) for every $B \in P$, $f(B) \in B$; and

(2) for every $B, C \in P$, either $f(B \cup C) = f(B)$ or $f(B \cup C) = f(C)$

How many choice functions are there?

Solution. The number of choice functions is the number of orderings of A, and so is $5! = 120$. To see this, consider an ordering π of A and observe that if $f(P)$ is defined to be the first member of P under π, then f is clearly a choice function. So we must now show that any choice function f arises from such an ordering. Let $n_1 = f(A)$. Now if $B \in P$ contains n_1, consider the decomposition of A into B and $A \setminus B$. By (2) we must have $f(A) = f(B)$ or $f(A) = f(A \setminus B)$, but the latter is not possible because n_1 is not in $A \setminus B$. So we have $f(B) = n_1$ for such B. Make n_1 the first member of the ordering that will define f. Now let $n_2 = f(A \setminus \{n_1\})$ and make the same argument: if n_2 is second in the ordering, then any set containing n_2 and not containing n_1 gets assigned n_2. Continuing through five steps yields the ordering $\pi = (n_1, n_2, n_3, n_4, n_5)$ and the choice function defined by "least in π" agrees with f.