Twenty-Sixth Annual Konhauser Problemfest

University of St. Thomas — February 17, 2018

Problems set by Răzvan Gelca, Texas Tech University

INSTRUCTIONS: Each team must hand in all work to be graded at the same time (at the end of the three-hour period). Each problem must be written on a separate page (or pages) and YOUR TEAM LETTER(S) SHOULD APPEAR AT THE TOP OF EVERY PAGE. Only one version of each problem will be accepted per team. Calculators of any sort are allowed. Justifications and/or explanations are expected for all problems. All ten problems will be weighted equally, and partial credit will be given for substantial progress toward a solution.

1. A point with coordinates (x, y) is chosen randomly in the triangle with vertices $(0, 0)$, $(0, 2\pi)$, $(2\pi, 2\pi)$. What is the probability that $\sin x > \sin y$?

2. Find, with proof, all real numbers x that satisfy the equation
$$\frac{27^x - 8^x}{19} = \frac{18^x - 12^x}{6}.$$

3. Consider the sequence defined recursively by
$$a_1 = \frac{4}{5}, \quad a_{n+1} = \frac{1}{5}\left(4a_n - 3\sqrt{1 - a_n^2}\right).$$
Prove that a_n is a rational number for all positive integers n.

4. Find, with proof, $\max_{|z|=1, |w|=2} \left| \frac{z}{w} - z - 2 \right|$, where z and w are complex.

5. The 52 students in a training program took a three-problem team selection test. Each student got at least one correct solution. There were 40 correct solutions on problem 1, 25 correct solutions on problem 2, and 18 correct solutions on problem 3. It is known that half of the students who solved problem 1 solved at least one of problems 2 and 3, all but 1 of the students who solved problem 2 solved at least one of problems 1 and 3, and two thirds of the students who solved problem 3 solved at least one of problems 1 and 2. How many students solved all 3 problems?

6. Find, with proof, all polynomials $P(x)$ with complex coefficients such that
$$P(x + 1)P(x - 1) = P(x^2 - 1).$$

7. Show that there is no permutation σ of the set $\{1, 2, \ldots, 2018\}$ such that

$$\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & \cdots & 2017 & 2018 \\ 2018 & 2017 & 2016 & \cdots & 2 & 1 \end{pmatrix},$$

where $\sigma^2 = \sigma \circ \sigma$.

8. Let $n \geq 2$ be a positive integer. Solve the equation

$$\begin{vmatrix} 1 & x & x^2 & x^3 & \cdots & x^{n-1} \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & 4 & \cdots & n \\ 1^2 & 2^2 & 3^2 & 4^2 & \cdots & n^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1^{n-2} & 2^{n-2} & 3^{n-2} & 4^{n-2} & \cdots & n^{n-2} \end{vmatrix} = 0.$$

9. Find all continuous functions $f : [0, \infty) \to \mathbb{R}$ that satisfy the equality

$$f(x^2) = \int_0^x \frac{f(t^2)}{t^2 + 1} dt$$

for all $x > 0$.

10. Let a_n be the number whose digits in the decimal expansion equal to the first n digits of the sequence obtained by periodically repeating the digits from 1 to 9:

$$123456789123456789123456789 12...$$

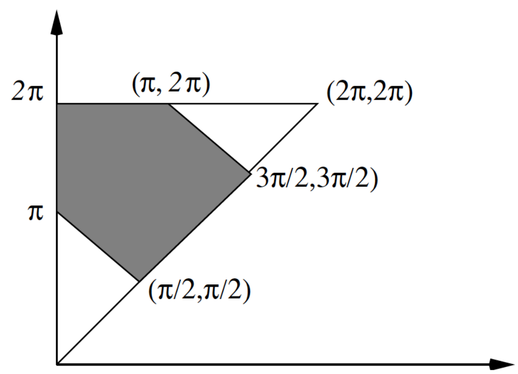
(for example $a_{12} = 123456789123$). Show that the set of prime divisors of the numbers a_n, $n \geq 1$, is infinite.

Konhauser 2018 - Solutions

by Răzvan Gelca, Texas Tech University
Updates: Doug Dokken, Kurt Scholz, Misha Shvartsman

1. A point of coordinates (x, y) is chosen randomly in the triangle with vertices $(0, 0)$, $(0, 2\pi)$, $(2\pi, 2\pi)$. What is the probability that $\sin x > \sin y$?

Solution: The domain $\sin x > \sin y$ is the shaded area from the figure. The entire triangle has area $2\pi^2$, the shaded region has area $\frac{3\pi^2}{2}$, hence the probability is $\frac{3}{4}$.



2. Find, with proof, all real numbers x that satisfy the equation

$$\frac{27^x - 8^x}{19} = \frac{18^x - 12^x}{6}.$$

Solution: Write the equation as $6 \cdot 3^{3x} - 19 \cdot 3^{2x}2^x + 19 \cdot 2^x 3^{2x} - 6 \cdot 2^{3x} = 0$, then divide by 2^{3x} and set $(3/2)^x = y$. The equation $6y^3 - 19y^2 + 19y - 6 = 0$ has solutions $y = 1, 3/2, 2/3$. So, the original equation has solutions $x = 0, x = 1, x = -1$.

3. Consider the sequence defined recursively by $a_1 = 4/5$, $a_{n+1} = \dfrac{1}{5}(4a_n - 3\sqrt{1 - a_n^2})$. Prove that a_n is a rational number for all positive integers n.

Solution: We can choose $0 < \theta < \pi/2$ so that $a_1 = \cos\theta = 4/5$, $\sin\theta = 3/5$, $a_2 = \cos 2\theta = 7/25$. We note that $\sin n\theta$ and $\cos n\theta$ are rational for any integer $n \geq 1$. Indeed, $\sin\theta$ and $\cos\theta$ are rational, and $\sin n\theta$ and $\cos n\theta$ can be expressed via trigonometric identities by additions, subtractions, and multiplications of $\sin\theta$ and $\cos\theta$. There is a lowest positive integer k such that $\sin k\theta < 0$. Specifically, $0.64 < \theta < 0.65$ in radians, $0 < 2.56 < 4\theta < 2.60 < \pi$ and $\pi < 3.20 < 5\theta < 3.25 < 3\pi/2$. So $\sin n\theta > 0$ for $1 \leq n \leq 4$ and $\sin 5\theta < 0$. Then, by induction, $a_n = \cos n\theta$, for $1 \leq n \leq 5$. Indeed, suppose $a_k = \cos k\theta$. Then $a_{k+1} = (4a_k - 3\sqrt{1 - a_k^2})/5 = \cos\theta\cos k\theta - \sin\theta\sin k\theta = \cos(k+1)\theta$. Furthermore, $a_6 = (4a_5 - 3\sqrt{1 - a_5^2})/5 = \cos 5\theta\cos\theta + \sin 5\theta\sin\theta = \cos 4\theta = a_4$. Thus, the sequence will cycle between 2 rational values $a_5 = \cos 5\theta$, $a_6 = a_4 = \cos 4\theta$, $a_7 = a_5 = \cos 5\theta$, and so on.

4. Find, with proof, $\max\limits_{|z|=1,|w|=2}\left|\dfrac{z}{w}-z-2\right|$, where z and w are complex.

Solution: For $|z|=1$, $|w|=2$, $\left|\dfrac{z}{w}-z-2\right|=\dfrac{1}{|w|}\left|z-zw-2w\right|=\dfrac{1}{2}\left|z(1-w)-2w\right|$ $\leq\dfrac{1}{2}\left(|z|\left|1-w\right|+2\left|w\right|\right)=\dfrac{1}{2}\left(|1-w|+4\right)\leq\dfrac{1}{2}\left(1+|w|+4\right)=\dfrac{7}{2}$. For $z=1$, $w=-2$, $\left|\dfrac{z}{w}-z-2\right|=\dfrac{7}{2}$. So, $\max\limits_{|z|=1,|w|=2}\left|\dfrac{z}{w}-z-2\right|=\dfrac{7}{2}$.

5. The 52 students in a training program took a three-problem team selection test. Each student got at least one correct solution. There were 40 correct solutions on problem 1, 25 correct solutions on problem 2, and 18 correct solutions on problem 3. It is known that half of the students who solved problem 1 solved at least one of problems 2 and 3, all but 1 of the students who solved problem 2 solved at least one of problems 1 and 3, and two thirds of the students who solved problem 3 solved at least one of problems 1 and 2. How many students solved all 3 problems?

Solution: From the statement we deduce that 20 students solved only problem 1, 1 student solved only problem 2, and 6 students solved only problem 3. So there are

$$52-\left(\frac{40}{2}+1+\frac{18}{3}\right)=25$$

students who solved at least 2 problems. Of these, 20 solved problems 1 and 2 or 1 and 3, 24 solved problems 2 and 1 or 2 and 3, and 12 solved problems 1 and 3 or 2 and 3. Among these are those students who solved all three problems.

So we have 25 students who solved at least two problems, and 20 of them solved 1 and 2 or 1 and 3 (and of course among these 20 there are some who might have solved all three problems). But from here we see that exactly $25-20=5$ solved only problems 2 and 3, and have not solved problem 1. Similarly $25-24=1$ solved only problems 1 and 3 and not problem 2, and $25-12=13$ solved only problems 1 and 2 but not problem 3. So there are $5+1+13=19$ students who solved exactly two problems.

We conclude that there were $25-19=6$ students who solved all three problems.

6. Find all polynomials $P(x)$ with complex coefficients such that $P(x+1)P(x-1)=P(x^2-1)$.

Solution: Clearly $P(x)\equiv 1$ and $P(x)\equiv 0$ are solutions, so let us assume that $P(x)$ is nonconstant. Note that if α is a zero of $P(x)$ then $(\alpha-1)^2-1=\alpha^2-2\alpha$ and $(\alpha+1)^2-1=\alpha^2+2\alpha$ are both zeros of $P(x)$. So the set S of zeros of $P(x)$ is a finite set that is closed under the transformations $\alpha\mapsto\alpha(\alpha-2)$ and $\alpha\mapsto\alpha(\alpha+2)$. We claim that the only set of complex numbers with this property is $S=\{0\}$. Indeed, let $\alpha\in S$ have maximal absolute value. Then $|\alpha||\alpha-2|\leq|\alpha|$ and $|\alpha||\alpha+2|\leq|\alpha|$. Because the disks $|\alpha-2|\leq 1$ and $|\alpha+2|\leq 1$ are disjoint, the two inequalities can hold simultaneously only if $\alpha=0$. Hence $P(x)=Cx^m$, $m=1,2,\ldots$. Since $C^2=C$ the nonzero polynomial will have $C=1$ and $P(x)=x^m$ which clearly satisfies the desired equality.

7. Show that there is no permutation σ of the set $\{1, 2, \dots, 2018\}$ such that

$$\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & \cdots & 2017 & 2018 \\ 2018 & 2017 & 2016 & \cdots & 2 & 1 \end{pmatrix},$$

where $\sigma^2 = \sigma \circ \sigma$.

Solution: The permutation

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & 2017 & 2018 \\ 2018 & 2017 & 2016 & \cdots & 2 & 1 \end{pmatrix}$$

has $2017 + 2016 + \cdots + 1 = \frac{2017 \times 2018}{2} = 1009 \times 2017$ inversions, so it is an odd permutation. On the other hand the signature is multiplicative so the signature of a square is the square of the signature, so it is positive. Hence the given permutation is not a square.

8. Let $n \geq 2$ be a positive integer. Solve the equation

$$\begin{vmatrix} 1 & x & x^2 & x^3 & \cdots & x^{n-1} \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & 4 & \cdots & n \\ 1^2 & 2^2 & 3^2 & 4^2 & \cdots & n^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1^{n-2} & 2^{n-2} & 3^{n-2} & 4^{n-2} & \cdots & n^{n-2} \end{vmatrix} = 0.$$

Solution: For every k, expanding $(n-1)(n-2)\dots(n-k)$ we can write n^k as a linear combination of $1, n, n^2, \dots, n^{k-1}, (n-1)(n-2)\dots(n-k)$, where the coefficients of the linear combination depend only on k and not on n. So, if we denote the original determinant by $f(x)$, for $1 \leq k \leq n-2$, the row $k+2$ of $f^{(k)}(1)$ is a linear combination of the first $k+1$ rows. For example, $k = 2$, $k + 2 = 4$, the first four rows of $f''(1)$ are $\{0, 0, 2, 6, 12, \dots, (n-1)(n-2)\}, \{1, 1, 1, \dots, 1\}, \{1, 2, 3, \dots, n\}, \{1^2, 2^2, 3^2, \dots, n^2\}$. The linear dependence of the first four rows can be seen by the expansion $(n-1)(n-2) = n^2 - 3n + 2$: the row "$n^2$" (4th row) minus 3 times the row "n" (3rd row) plus 2 times the $1's$ row (2nd row) is equal to the 1st row. This shows that for $1 \leq k \leq n-2$, $f^k(1) = 0$, which combined with $f(1) = 0$ implies that the Taylor series of f at 1 is of the form $a(x-1)^{n-1}$. So the given equation has the unique solution $x = 1$.

9. Find all continuous functions $f : [0, \infty) \to \mathbb{R}$ that satisfy the equality

$$f(x^2) = \int_0^x \frac{f(t^2)}{t^2 + 1} dt,$$

for all $x > 0$.

Solution: The right-hand side is differentiable with respect to x, so the left-hand side is differentiable, too. Since $f(x^2)$ is differentiable, so is $f(x)$ (as we compose with $f(x^2)$

with $\sqrt{(x)}$). Differentiating we obtain $f'(x^2)2x = \frac{f(x^2)}{x^2+1}$, $x > 0$. Assume that there is an interval (a, b) on which f is not equal to 0, and choose this interval to be maximal. Then for $x \in (\sqrt{a}, \sqrt{b})$, $\frac{f'(x^2)2x}{f(x^2)} = \frac{1}{x^2+1}$, and integrating we obtain that on this interval, $\ln f(x^2) = \int \frac{1}{x^2+1} dx = \arctan x + c_0$. Letting $x \to \sqrt{a}$, we conclude that $\ln f(a) \in \mathbb{R}$, so $f(a) \neq 0$. Note that $a > 0$, because $f(0) = \lim_{x \to 0^+} \int_0^x \frac{f(t^2)}{t^2+1} dt = 0$. Because of the continuity of f, there is an interval $(a-\epsilon, a+\epsilon)$ on which f in nonzero, hence f is nonzero on $(a-\epsilon, b)$, which contradicts the maximality of (a, b). This contradiction shows that f must be zero everywhere. Thus the only solution is $f(x) = 0$ for all x. As a remark, the solutions to the original equation do not necessarily coincide with the solutions to $f'(x^2)2x = \frac{f(x^2)}{x^2+1}$. By differentiation we introducted additional solutions, which have to be substituted in the original equation in order to check whether they are actual solutions or not.

10. Let a_n be the number whose digits in the decimal expansion equal to the first n digits of the sequence obtained by periodically repeating the digits from 1 to 9:

$$12345678912345678912345678912...$$

(for example $a_{12} = 123456789123$). Show that the set of prime divisors of the numbers a_n, $n \geq 1$, is infinite.

Solution: We will show that the set of prime divisors of the numbers a_{9n}, $n \geq 1$ is infinite. Note that $a_{9n} = 123456789 \times 1000000001000000001...001$, where in the last factor there are n ones, and between any consecutive ones there are 8 zeros. We follow Euclid's proof for the existence of infinitely many primes and, by arguing by contradiction, we assume that the set of prime divisors of a_{9n}, $n \geq 1$, is finite. Then the set of prime divisors of the numbers

$$b_n = \frac{10^{9n}-1}{10^9-1} = 1000000001000000001...001, \quad n \geq 1$$

is finite; let these prime divisors be $p_1 = 3 < p_2 < \ldots < p_m$. Assume that p_k divides some b_{n_k} for $k = 1, 2, ..., m$. If we let $N = n_1 n_2 \cdots n_k$ then b_N is a multiple of b_{n_k} for all $k = 1, 2, ..., m$. This is because for any $r, s \in \mathbb{N}$, b_{rs} is obtained by repeating the digits of b_r s times in a sequence, so $b_{rs} = b_r \times \frac{10^{10rs}-1}{10^{10r}-1}$. Hence p_k divides b_N for all $k = 1, 2, ..., m$. This implies that $b_{N+1} = 10^9 b_N + 1$ is coprime with b_N so it is not divisible by any of p_k, $k = 1, 2, ..., m$. This is a contradiction, which implies the desired conclusion.