Twenty-Seventh Annual Konhauser Problemfest

St. Olaf College
February 23, 2019
problems by Jeremy Rouse (Wake Forest University)

Instructions: Each team must hand in all work to be graded at the end of the three-hour period. Each problem must be written on a separate page (or pages), and **the team identifier should appear at the top of each page**. Only one solution for each problem will be accepted per team. Calculators of any sort are allowed. Justifications and/or explanations are expected for all problems. All ten problems will be weighted equally, and partial credit will be given for substantial progress toward a solution.

1. Group projects

A professor is teaching a projective geometry class with 13 students in it. She wishes for the students (numbered from 1 to 13) to complete 13 group projects, where four students will work together on each project. The assignments must satisfy the following additional rules:

(a) Each pair of students is assigned to work on exactly one project together.

(b) If i and j are any two numbers between 1 and 13 (inclusive), then student i works on project j if and only if student j works on project i.

(c) The professor has already assigned students to projects 1, 4, 7, 10, and 13.

Find a valid assignment of students to the remaining eight projects, and record those assignments in the table below.

Project	Students working on project
1	2,3,4,5
2	
3	
4	1,5,10,11
5	
6	
7	2,6,10,12
8	
9	
10	4,7,8,10
11	
12	
13	5,6,8,13

2. **Biggest cone you can fit inside a watermelon**

The great watermelon has the shape of an ellipsoid, and is given by the equation

$$\frac{x^2}{9^2} + \frac{y^2}{3^2} + \frac{z^2}{3^2} = 1.$$

What is the largest volume a right circular cone can have that fits inside the great watermelon?

3. **Which number of dominoes is most likely?**

There are 10946 ways that a 1×20 rectangle can be tiled with squares (that are 1×1) and dominoes (that are 1×2). All 10946 such tilings are put in the Fibonacci hat. If you randomly pick a tiling out of the hat, is it more likely that it will have

(a) exactly 5 dominoes (and hence 10 squares), or

(b) exactly 6 dominoes (and hence 8 squares)?

4. **Tricky tangent**

Consider the curve parametrized by $x(t) = t^3 + 3t^2 + 3t$ and $y(t) = t^4 - 6t^2 - 8t$ for $-2 < t < 2$. For this range of t values, y can be considered as a continuously differentiable function of x. Compute $\frac{dy}{dx}$ at the point where $t = -1$ (so $x = -1$ and $y = 3$).

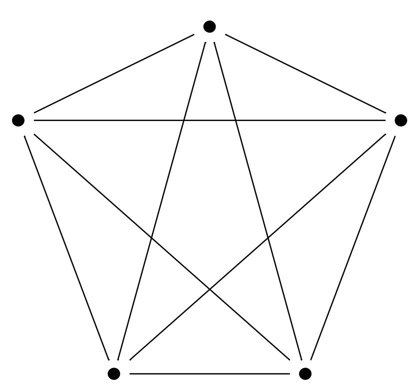
5. **An odd reciprocal limit**

Evaluate

$$\lim_{n\to\infty} \left(\frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n-1} \right).$$

6. **Red and blue triangles**

Ralph and Ben play a game. They start with a drawing of the sides and diagonals of a pentagon.



Ralph plays first, and on his turn, he colors one edge red. Ben plays next and on his turn he colors one edge blue. The players take turns (coloring one edge at a time) until all ten edges are colored. At this point, Ben wins if the colored diagram contains either a red triangle or a blue triangle. If there is no such triangle, then Ralph wins.

Describe a winning strategy for one of the two players.

7. Positive polynomial problem

Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with real coefficients, and assume that $p(x) \geq p'(x)$ for all real values of x. Prove that $p(x) \geq 0$ for all x.

8. Matrices with integer square roots

Let $A = \begin{bmatrix} 0 & 4 \\ 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & -9 \\ 0 & 0 \end{bmatrix}$. Prove that for all positive integers n, there is a matrix C_n with integer entries so that $C_n^2 = A^n + B^n$.

9. An orthogonal integer basis

Let $\vec{v}_1, \ldots, \vec{v}_n$ be a set of n orthogonal vectors in $\mathbb{R}^n$. Assume that all the entries of each $\vec{v}_i$ are integers. Prove that the product of the lengths, $\|\vec{v}_1\|\|\vec{v}_2\| \cdots \|\vec{v}_n\|$, is an integer.

10. Superpowers of 3

Prove that for every positive integer n, there are integers x and y so that $3^{3^n} - 1 = x^2 + y^2$. (That is, show that $3^3 - 1$, $3^9 - 1$, $3^{27} - 1$, $3^{81} - 1$, etc. can each be written as the sum of two perfect squares.)

2019 Konhauser Solutions

by Jeremy Rouse (Wake Forest University)

1. **Group projects**

A professor is teaching a projective geometry class with 13 students in it. She wishes for the students (numbered from 1 to 13) to complete 13 group projects, where four students will work together on each project. The assignments must satisfy the following additional rules:

(a) Each pair of students is assigned to work on exactly one project together.

(b) If i and j are any two numbers between 1 and 13 (inclusive), then student i works on project j if and only if student j works on project i.

(c) The professor has already assigned students to projects 1, 4, 7, 10, and 13.

Find a valid assignment of students to the remaining eight projects, and record those assignments in the table below.

Project	Students working on project
1	2,3,4,5
2	
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4	1,5,10,11
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9	
10	4,7,8,10
11	
12	
13	5,6,8,13

Solution: Since we're given the students that work on projects 1, 4, 7, 10 and 13, by property (b), we also know which projects students 1, 4, 7, 10 an 13 work on.

We see now that student 10 is already assigned to work with everyone except students 3 and 9, so these must be the last two students to work on project 8. Thus, student 8 works on projects 3 and 9.

We have that students 1, 4 and 13 work on project 5. Using property (a), we see that the last student to work on project 5 cannot be 2, 3 or 5 (because of project 1), 6 (because of project 13), 7 (because of project 6), 8 (because of project 10), 9 (because of project 8), 10 (because of project 8), or 13 (because of project 13). Also, student 11 cannot work on project 5, since then student 5 would work on project 11, but student 5 has already been assigned to work with student 4 on the first project. Hence the last student to work on project 5 must be student 12, and so student 5 must work on project 12. Student 5 is now assigned to work with everyone except students 9 and 12, and so these are the last two students to work on project 12.

Student 9 must work with student 1 on either project 2 or 3, but student 9 cannot work on project 2, since they work on project 12 with student 7. Thus they must work on project 3. Thus, student 3 cannot work on project 3, so student 3 (who must work with 1 sometime) must work on project 2. Hence student 2 works on project 3. The last student to work on project 2 must be student 6. (And so student 2 works on project 6.)

Student 3 is now assigned to work with everyone except student 11, so student 11 works on project 9 (and student 9 works on project 11).

Everything is now assigned except for the last student on project 6 and the last student on project 11. By symmetry, student 6 and 11 are only assigned to three projects, and so student 11 must work on project 6, and student 6 on project 11.

Project	Students working on project
1	2,3,4,5
2	1,3,6,7
3	1,2,8,9
4	1,5,10,11
5	1,4,12,13
6	2,7,11,13
7	2,6,10,12
8	3,9,10,13
9	3,8,11,12
10	4,7,8,10
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Commentary: The solution is unique. The problem is connected with projective geometry, as the source of this problem is a self-dual Steiner triple system. This triple system comes from the points/lines in the projective plane over the field $\mathbb{F}_3$. There are four points on each line in $\mathbb{P}^2(\mathbb{F}_3)$, and each point is contained in four lines. Any two points are contained in a unique line.

2. Biggest cone you can fit inside a watermelon

The great watermelon has the shape of an ellipsoid, and is given by the equation

$$\frac{x^2}{9^2} + \frac{y^2}{3^2} + \frac{z^2}{3^2} = 1.$$

What is the largest volume a right circular cone can have that fits inside the great watermelon?

Solution: In order for the cone to have a circular base, it must be inscribed in the ellipsoid on a plane of the form $x = x_0$. Suppose that one vertex of the cone is placed at $(-9, 0, 0)$ and the base of the cone is in the plane $x = x_0$. If r is the radius of the base, then $y^2 + z^2 = r^2$ and so $1 - \frac{x_0^2}{9^2} = r^2/9$ and thus $r^2 = 9 - \frac{x_0^2}{9}$. The volume of the cone is $\frac{1}{3}\pi r^2 h$ and $h = x_0 + 9$. Thus, we wish to maximize

$$V(x_0) = \frac{1}{3}\pi(x_0 + 9) \cdot \left(9 - \frac{x_0^2}{9}\right) = \pi\left(27 + 3x_0 - x_0^2/3 - \frac{x_0^3}{27}\right)$$

for $x_0 \in [-9, 9]$. It's easy to see that $V(-9) = V(9) = 0$, and so the maximum occurs at a critical point.

We find that $V'(x_0) = \pi\left(3 - 2/3x_0 - \frac{x_0^2}{9}\right)$. Setting this to zero gives $27 - 6x_0 - x_0^2 = 0$ or $x_0^2 + 6x_0 - 27 = 0$. We get $x_0 = \frac{-6\pm\sqrt{36+4\cdot1\cdot27}}{2} = \frac{-6\pm\sqrt{36+108}}{2} = \frac{-6\pm\sqrt{144}}{2}$ and so $x_0 = -9$ or $x_0 = 3$. The maximum must occur when $x_0 = 3$ and the volume when $x_0 = 3$ is

$$V(3) = \frac{1}{3}\pi(12) \cdot (9 - 1) = 32\pi.$$

3. Which number of dominoes is most likely?

There are 10946 ways that a 1×20 rectangle can be tiled with squares (that are 1×1) and dominoes (that are 1×2). All 10946 such tilings are put in the Fibonacci hat. If you randomly pick a tiling out of the hat, is it more likely that it will have

(a) exactly 5 dominoes (and hence 10 squares), or

(b) exactly 6 dominoes (and hence 8 squares)?

Solution: The probability of drawing a tiling with exactly 5 dominoes is *the same* as the probability of drawing a tiling with exactly 6 dominoes.

The number of total tilings is 10946. To create a tiling with 5 dominoes, there must be $10 + 5 = 15$ total tiles, and we must choose which 5 tiles out of those 15 to make dominoes. So the number of ways to do this is the number of ways to choose a 5-element subset of $\{1, 2, \ldots, 15\}$ and this is $\binom{15}{5} = \frac{15\cdot14\cdot13\cdot12\cdot11}{1\cdot2\cdot3\cdot4\cdot5}$. We can cancel the 3 and 5 with the 15, we can divide the 12 by 4 and we can divide the 14 by 2. We get $7 \cdot 13 \cdot 3 \cdot 11 = 91 \cdot 33 = 33 + 2970 = 3003$.

To create a tiling with 6 dominoes, there must be $8 + 6 = 14$ total tiles, and we must choose a 6-element subset of those tiles to be dominoes. There are $\binom{14}{6} = \frac{14\cdot13\cdot12\cdot11\cdot10\cdot9}{1\cdot2\cdot3\cdot4\cdot5\cdot6}$

ways to do this. We can cancel 12 with $3 \cdot 4$ and 10 with $2 \cdot 5$. We're left with

$$\frac{14 \cdot 13 \cdot 11 \cdot 9}{6} = 7 \cdot 13 \cdot 11 \cdot 3 = 91 \cdot 33 = 3003$$

again.

4. Tricky tangent

Consider the curve parametrized by $x(t) = t^3 + 3t^2 + 3t$ and $y(t) = t^4 - 6t^2 - 8t$ for $-2 < t < 2$. For this range of t values, y can be considered as a continuously differentiable function of x. Compute $\frac{dy}{dx}$ at the point where $t = -1$ (so $x = -1$ and $y = 3$).

Solution: If t is a real number so that $x'(t) \neq 0$, then at the point $(x(t), y(t))$ we have $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$. We have $y'(t) = 4t^3 - 12t - 8$ and $x'(t) = 3t^2 + 6t + 3 = 3(t+1)^2$. When one plugs in $t = -1$, we obtain that $x'(t) = 0$. However, we are given that y is a continuously differential function of x and so

$$\begin{aligned} \frac{dy}{dx}|_{x=-1} &= \lim_{t \to -1} \frac{y'(t)}{x'(t)} \\ &= \lim_{t \to -1} \frac{4t^3 - 12t - 8}{3(t+1)^2}. \end{aligned}$$

We can factor $4t^3 - 12t - 8 = (t+1)(4t^2 - 4t - 8)$ and $4t^2 - 4t - 8 = (t+1)(4t - 8)$. So our limit above is

$$\lim_{t \to -1} \frac{(t+1)^2(4t-8)}{3(t+1)^2} = \lim_{t \to -1} \frac{4t-8}{3} = (-4-8)/3 = -12/3 = -4.$$

Commentary: The students must be careful to use that $\frac{dy}{dx}$ is continuously differentiable to avoid dividing by zero.

5. An odd reciprocal limit

Evaluate

$$\lim_{n \to \infty} \left(\frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n-1} \right).$$

Solution: We will write down a Riemann sum for $\int_2^4 \frac{dx}{2x}$ and we will choose the point $x_i^* \in [x_{i-1}, x_i]$ to be the midpoint. We have $\Delta x = \frac{4-2}{n} = \frac{2}{n}$. We have $x_i = 2 + i\Delta x = 2 + \frac{2i}{n}$ and we have $x_i^* = \frac{x_{i-1}+x_i}{2} = \frac{2+2(i-1)/n+2+2i/n}{2} = 2 + (2i-1)/2n$. The Riemann sum is

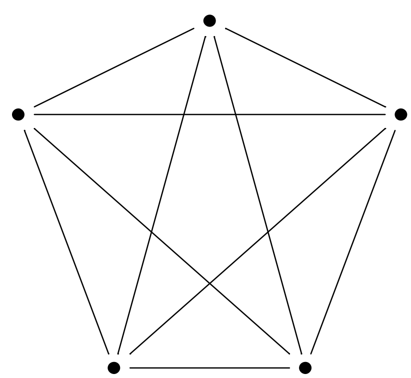
$$\begin{aligned} \sum_{i=1}^{n} f(x_i^*)\Delta x &= \sum_{i=1}^{n} f\left(2 + (2i-1)/n\right) \cdot \frac{1}{n} \\ &= \sum_{i=1}^{n} \frac{1/2}{2 + (2i-1)/n} \cdot \frac{2}{n} = \sum_{i=1}^{n} \frac{1}{2 + (2i-1)/n}. \end{aligned}$$

This is precisely the sum that is above. As $n \to \infty$, the Riemann sum tends to the integral and so the limit is $\int_2^4 \frac{dx}{2x} = \frac{1}{2}[\ln(x)]_2^4 = \frac{1}{2}\ln(4) - \frac{1}{2}\ln(2) = \ln(2) - \frac{1}{2}\ln(2) = \frac{1}{2}\ln(2)$.

Commentary: Another way to do the problem is to define $H_n = 1 + 1/2 + \cdots + 1/n$ and to know that $H_n \sim \ln(n) + \gamma$, where $\gamma \approx 0.5772$. The sum above is $(H_{4n} - H_{2n}) - \frac{1}{2}(H_{2n} - H_n)$ and the result follows from this.

6. Red and blue triangles

Ralph and Ben play a game. They start with a drawing of the sides and diagonals of a pentagon.



Ralph plays first, and on his turn, he colors one edge red. Ben plays next and on his turn he colors one edge blue. The players take turns (coloring one edge at a time) until all ten edges are colored. At this point, Ben wins if the colored diagram contains either a red triangle or a blue triangle. If there is no such triangle, then Ralph wins.

Describe a winning strategy for one of the two players.

Solution: Ben has a winning strategy. Note that if Ben can arrange to have one vertex of the pentagon to have three edges that are the same color coming out of it, he will win. For example, if v is a vertex and three edges coming out of v (going to v_1, v_2 and v_3) are red, then if any edge $v_i - v_j$ is red, there will be a red triangle. If all three of the edges connecting v to v_i are blue, then $v_1 - v_2 - v_3 - v_1$ is a blue triangle.

Ralph starts by drawing an edge red. Ben draws an edge blue that doesn't share a vertex with the edge colored red. On his next move, Ralph colors another edge red, but Ralph can only draw an edge coming out of the two vertices of the blue edge. Ben draws another blue edge coming out of a vertex of the first blue edge, but he avoids a vertex that has a red edge coming out of it. At this point, one of the vertices has two blue edges coming out of it, and no red edges. Since there are a total of four edges coming out of this point, on his next move (regardless of what Ralph does), Ben can draw a third blue edge, guaranteeing that Ben will win.

Commentary: This is a Ramsey-theory-ish problem. It's variant of the reasonably well-known fact that if you color every edge of a complete graph K_6 red or blue, there must either be a red triangle or a blue triangle. It's not true that every red-blue coloring of the edges and diagonals of K_5 has a triangle, but the only way to avoid that is to have the K_5 be the union of a red pentagon and a blue pentagon.

7. **Positive polynomial problem**

Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with real coefficients, and assume that $p(x) \geq p'(x)$ for all real values of x. Prove that $p(x) \geq 0$ for all x.

Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with real coefficients, and assume that $p(x) \geq p'(x)$ for all real values of x. Prove that $p(x) \geq 0$ for all x.

Solution: First, observe that the leading coefficient of $p(x)$ must be positive. (If it was negative, then because $p(x)$ grows faster than $p'(x)$ we would have $p(x) < p'(x)$ for large x.) Next, the degree of $p(x)$ must be even, because if $p(x)$ has odd degree, then $p(x) \to -\infty$ as $x \to -\infty$, but $p'(x) \to \infty$ as $x \to -\infty$.

Since $p(x)$ has even degree and positive leading coefficient $p(x) \to \infty$ as $x \to \infty$ and $p(x) \to \infty$ as $x \to -\infty$. It follows that there is some $x_0 \in \mathbb{R}$ so that $p(x) \geq p(x_0)$ for all $x \in \mathbb{R}$. This $p(x_0)$ is an absolute minimum, and so we must have $p'(x_0) = 0$. We then have that $p(x_0) \geq p'(x_0) = 0$ and so for all x, $p(x) \geq p(x_0) \geq p'(x_0) = 0$, proving the claim.

8. **Matrices with integer square roots**

Let $A = \begin{bmatrix} 0 & 4 \\ 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & -9 \\ 0 & 0 \end{bmatrix}$. Prove that for all positive integers n, there is a matrix C_n with integer entries so that $C_n^2 = A^n + B^n$.

Solution 1: Note that

$$AB = \begin{bmatrix} 0 & 4 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 9 & -9 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 9 & -9 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 36 - 36 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Using these facts, we claim that $(A^n + B^n) = (A + B)^n$. We prove this by induction on n, and the claim is clearly true for $n = 1$. Assuming that $(A^{n-1} + B^{n-1}) = (A + B)^{n-1}$, we have

$$\begin{aligned}(A + B)^n &= (A + B)^{n-1}(A + B) = (A^{n-1} + B^{n-1})(A + B) \\ &= A^{n-1}A + B^{n-1}A + A^{n-1}B + B^{n-1}B \\ &= A^n + B^{n-2}(BA) + A^{n-2}(AB) + B^n = A^n + B^n\end{aligned}$$

since $AB = BA = 0$. This proves the claim. Hence, it suffices to show that there is a matrix C with integer entries so that $C^2 = A + B$. Then $A^n + B^n = (A + B)^n = (C^2)^n = (C^n)^2$, and we may take $C_n = C^n$.

We have $A + B = \begin{bmatrix} 9 & -5 \\ 0 & 4 \end{bmatrix}$. If $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$C^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}.$$

Since $ac + cd = c(a+d) = 0$, we either have $c = 0$ or $a + d = 0$. If $a + d = 0$ then $a = -d$ and this forces $a^2 + bc = 9$ and $bc + d^2 = bc + a^2 = 4$. This is a contradiction.

Thus, $c = 0$ and we get $a^2 = 9$, $d^2 = 4$ and $b(a+d) = -5$. There are four solutions to this system of equations corresponding to the matrices $C = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} -3 & 5 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 3 & -5 \\ 0 & -2 \end{bmatrix}$, and $\begin{bmatrix} -3 & 1 \\ 0 & -2 \end{bmatrix}$.

Solution 2: Another strategy to try here is to prove by induction that $A^n = \begin{bmatrix} 0 & 4^n \\ 0 & 4^n \end{bmatrix}$, $B^n = \begin{bmatrix} 9^n & -9^n \\ 0 & 0 \end{bmatrix}$. This is clear for $n = 1$. If we assume it's true for $n - 1$, then

$$\begin{aligned} A^n = AA^{n-1} &= \begin{bmatrix} 0 & 4 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 4^{n-1} \\ 0 & 4^{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 4^n \\ 0 & 4^n \end{bmatrix} \\ B^n = BB^{n-1} &= \begin{bmatrix} 9 & -9 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 9^{n-1} & -9^{n-1} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 9^n & -9^n \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Thus, $A^n + B^n = \begin{bmatrix} 9^n & 4^n - 9^n \\ 0 & 4^n \end{bmatrix}$. Again if we have $C_n = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ this leads to $a^2 + bc = 9^n$, $ab + bd = 4^n - 9^n$, $ac + cd = 0$ and $bc + d^2 = 4^n$. We must have $c(a + d) = 0$ so $c = 0$ or $a + d = 0$. If $a + d = 0$ then $d = -a$ and this contradicts $bc + a^2 = 9^n$ and $bc + d^2 = 4^n$. Thus, $c = 0$.

This gives $a^2 = 9^n$, $b(a+d) = 4^n - 9^n$ and $d^2 = 4^n$. Thus, $a = \pm 3^n$, $d = \pm 2^n$. Since $4^n - 9^n = (2^n + 3^n)(2^n - 3^n)$ any choice of the plus and minus signs for a and d work.

Commentary: Another way to do this problem is to diagonalize A and B. One can see that A and B have the same eigenvectors, and the diagonal matrix for A is $\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$ and the diagonal matrix for B is $\begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix}$. One can then easily construct C as the matrix with the same eigenvectors and eigenvalues 2 and 3. (I don't think I've seen a problem like this one before.)

9. An orthogonal integer basis

Let $\vec{v}_1, \ldots, \vec{v}_n$ be a set of n orthogonal vectors in $\mathbb{R}^n$. Assume that all the entries of each $\vec{v}_i$ are integers. Prove that the product of the lengths, $\|\vec{v}_1\|\|\vec{v}_2\| \cdots \|\vec{v}_n\|$, is an integer.

Solution: Let A be the $n \times n$ matrix whose columns are the $\vec{v}_1$, …, $\vec{v}_n$. Then,

$$A^TA = \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vdots \\ \vec{v}_n^T \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} \vec{v}_1^T\vec{v}_1 & \vec{v}_1^T\vec{v}_2 & \cdots & \vec{v}_1^T\vec{v}_n \\ \vec{v}_2^T\vec{v}_1 & \vec{v}_2^T\vec{v}_2 & \cdots & \vec{v}_2^T\vec{v}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{v}_n^T\vec{v}_1 & \vec{v}_n^T\vec{v}_2 & \cdots & \vec{v}_n^T\vec{v}_n \end{bmatrix}.$$

Now, $\vec{v}_i^T\vec{v}_j = \vec{v}_i \cdot \vec{v}_j$. Since the set of n vectors is orthogonal, the matrix A^TA is diagonal. Hence,

$$\begin{aligned}
\det(A^TA) &= (\vec{v}_1^T\vec{v}_1)(\vec{v}_2^T\vec{v}_2)\cdots(\vec{v}_n^T\vec{v}_n) \\
\det(A^T)\det(A) &= \|\vec{v}_1\|^2\|\vec{v}_2\|^2\cdots\|\vec{v}_n\|^2 \\
\det(A)^2 &= \left(\|\vec{v}_1\|\|\vec{v}_2\|\cdots\|\vec{v}_n\|\right)^2.
\end{aligned}$$

Since A has integer entries, $\det(A)$ is an integer. Thus,

$$\|\vec{v}_1\|\cdots\|\vec{v}_n\| = \pm\det(A)$$

is an integer.

10. Superpowers of 3

Prove that for every positive integer n, there are integers x and y so that $3^{3^n} - 1 = x^2 + y^2$. (That is, show that $3^3 - 1$, $3^9 - 1$, $3^{27} - 1$, $3^{81} - 1$, etc. can each be written as the sum of two perfect squares.)

Solution: We will prove the statement by induction on n. For the base case of $n = 1$, we have $3^3 - 1 = 26 = 5^2 + 1^2$. We will need to use that if a and b are each the sum of two squares, their product is the sum of two squares. This follows from the identity

$$\begin{aligned}
(c^2+d^2)(e^2+f^2) &= c^2e^2 + c^2f^2 + d^2e^2 + d^2f^2 \\
&= c^2e^2 + d^2f^2 + c^2f^2 + d^2e^2 = c^2e^2 + 2cdef + d^2f^2 + c^2f^2 - 2cdef + d^2e^2 \\
&= (ce+df)^2 + (cf-de)^2.
\end{aligned}$$

Now, assume that $3^{3^{n-1}} - 1$ is a sum of two squares. Using the identity $x^3 - 1 = (x-1)(x^2+x+1)$ with $x = 3^{3^{n-1}}$, we get

$$3^{3^n} - 1 = \left(3^{3^{n-1}} - 1\right)\left(3^{2\cdot 3^{n-1}} + 3^{3^{n-1}} + 1\right).$$

By the identity above, it suffices to prove that the second factor is a sum of two squares. We may write the second factor as

$$3^{2\cdot 3^{n-1}} - 2\cdot 3^{3^{n-1}} + 1 + 3\cdot 3^{3^{n-1}} = (3^{3^{n-1}} - 1)^2 + 3^{3^{n-1}+1} = (3^{3^{n-1}} - 1)^2 + \left(3^{\frac{3^{n-1}+1}{2}}\right)^2.$$

This proves that $3^{3^n} - 1$ is the product of two numbers that are each the sum of two squares, so it must be the sum of two squares.