Twenty-Eighth Annual Konhauser Problemfest

February 29, 2020, Macalester College
Problems set by Ian Whitehead (Swarthmore College)

1. **Circle in a Triangle.** A circle of radius 1 is inscribed in a 30-60-90 right triangle, with each side of the triangle tangent to the circle. Find the area of the triangle.

2. **2020 Vision.** The number 2020 has the following interesting property. Its prime power factors are $4 = 2^2$, 5, and 101. Each of these prime power factors is congruent to 1 modulo all the smaller factors, i.e.

$$\begin{aligned} 5 &\equiv 1 \mod 4 \\ 101 &\equiv 1 \mod 4 \\ 101 &\equiv 1 \mod 5. \end{aligned}$$

   Any positve integer n factors uniquely into prime powers $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ where the p_i are distinct primes. Find the largest integer which has the same property as 2020 and having all its prime power factors $p_j^{r_j}$ less than 100.

3. **Orange Peel.** A spherical orange of fixed radius intersects two parallel planes. Show that the surface area of orange peel between the planes depends on the distance between them, but does not depend on their location.

4. **Mashed Potatoes.** A group of n people sit around a table with plates of mashed potatoes in front of them. They all have an equal amount of mashed potatoes except for one person who has slightly more. In order to be more fair, each person puts exactly $\frac{1}{2}$ of their potatoes onto the plate of the person to their right. Then they repeat this process with the potatoes now on their plates, and repeat it again infinitely many times. Prove that each person at the table has more potatoes than anyone else infinitely many times.

5. **Closure.** Classify all sets S of positive integers with the following properties:

   - If $x \in S$ then $x + 13 \in S$.
   - If $x \in S$ and x is even then $\frac{x}{2} \in S$.

6. **A Complex Sum.** Find

$$\lim_{n\to\infty} \sum_{k=-n}^{n} \frac{1}{n+ik}.$$

   Here $i = \sqrt{-1}$.

7. **A Real Integral.** Suppose that $f(x)$ is the unique solution to the differential equation
$$f(x) - f'(x) = \sec(x)$$
such that $f(0) = 0$. Find
$$\int_0^{\pi/4} f(x)\sin(x)\,dx$$

8. **Nested Shapes.** A regular dodecahedron (12 pentagonal faces, 20 vertices) is inscribed in a regular icosahedron (20 triangular faces, 12 vertices), which is inscribed in a regular dodecahedron. The ratio of side lengths between the inner and outer dodecahedra is A. A regular icosahedron is inscribed in a regular dodecahedron, which is inscribed in a regular icosahedron. The ratio of side lengths between the inner and outer icosahedra is B. Prove one of the following claims: $A > B$, $A = B$, $A < B$. (Note: when one polyhedron is inscribed in another, the vertices of the inner polyhedron lie at the centers of the faces of the outer one.)

9. **Fix this Problem.** The following problem was intended to be on the 2020 Konhauser, but when we received it, the ink was smudged on a crucial piece of information.

   "Find positive integers a, b, c, d, e, f such that
$$\begin{aligned}&\operatorname{lcm}(a,b,c) = 60, \quad \operatorname{lcm}(b,c,d) = 540, \quad \operatorname{lcm}(c,d,e) = 135,\\ &\operatorname{lcm}(d,e,f) = 5454, \quad \operatorname{lcm}(e,f,a) = 1212, \quad \operatorname{lcm}(f,a,b) = \text{(illegible)}.\end{aligned}$$
   Here lcm denotes the least common multiple of a set of integers."

   The problem was intended to have a unique solution consisting of distinct positive integers a, b, c, d, e, f. Find the illegible number, and find (a,b,c,d,e,f).

10. **A Curious Matrix.** The function $\cos(\frac{2\pi}{n}x)$ is applied to the entries of an $n \times n$ multiplication table. The 3×3 example is shown below:
$$\begin{pmatrix} \cos(\frac{2\pi}{3}\cdot 1) & \cos(\frac{2\pi}{3}\cdot 2) & \cos(\frac{2\pi}{3}\cdot 3) \\ \cos(\frac{2\pi}{3}\cdot 2) & \cos(\frac{2\pi}{3}\cdot 4) & \cos(\frac{2\pi}{3}\cdot 6) \\ \cos(\frac{2\pi}{3}\cdot 3) & \cos(\frac{2\pi}{3}\cdot 6) & \cos(\frac{2\pi}{3}\cdot 9) \end{pmatrix}$$
    Show that any eigenvalue of this $n \times n$ matrix is either 0 or $\pm\sqrt{n}$.

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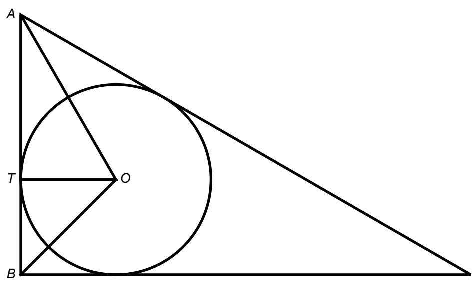
1. **Circle in a Triangle.** A circle of radius 1 is inscribed in a 30-60-90 right triangle, with each side of the triangle tangent to the circle. Find the area of the triangle.

   The area of the triangle is $3 + 2\sqrt{3}$. Connect the center of the circle O to the two vertices A, B, and the point of tangency T on the short side of the triangle as shown:

   We see that OAT and OBT are right triangles. Moreover, $\angle OAT$ bisects a $60°$ angle, so OAT is a 30-60-90 right triangle. And $\angle OBT$ bisects a $90°$ angle, so OBT is a 45-45-90 right triangle. Since $|OT| = 1$, we conclude that $|AT| = \sqrt{3}$ and $|BT| = 1$, so $|AB| = 1 + \sqrt{3}$. Then the area of the triangle is $\frac{\sqrt{3}}{2}|AB|^2 = 3 + 2\sqrt{3}$.

2. **2020 Vision.** The number 2020 has the following interesting property. Its prime power factors are $4 = 2^2$, 5, and 101. Each of these prime power factors is congruent to 1 modulo all the smaller factors, i.e.

   $$\begin{aligned} 5 &\equiv 1 \mod 4 \\ 101 &\equiv 1 \mod 4 \\ 101 &\equiv 1 \mod 5. \end{aligned}$$

   Any positve integer n factors uniquely into prime powers $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ where the p_i are distinct primes. Find the largest integer which has the same property as 2020 and having all its prime power factors $p_j^{r_j}$ less than 100.

   The number is 6806. Let q be the largest prime power factor of our desired number n. Then every other prime power factor of n must divide $q - 1$. Since these other prime power factors are coprime, their product must divide $q - 1$. So we see that n is at most $q(q - 1)$. Therefore it is advantageous to choose q as large as possible. For $q = 97$, the best we can do is $n = 4656$ with prime power factors $3, 16, 97$. For $q = 89$, the best we can do is $n = 1958$, with prime power factors $2, 11, 89$. For $q = 83$, we can get $n = q(q - 1) = 6806$, with prime power factors $2, 41, 83$. This is optimal and cannot be exceeded with any smaller choice of q. Indeed, the next largest prime is 79, and $79 \cdot 78 = 6162 < 6806$.

3. **Orange Peel.** A spherical orange of fixed radius intersects two parallel planes. Show that the surface area of orange peel between the planes depends on the distance between them, but does not depend on their location.

   Assume for simplicity that the sphere has radius 1. Also assume without loss of generality that the sphere is centered at the origin and that the two planes are parallel to the yz plane, at $x = a$ and $x = b$, for $-1 \leq a < b \leq 1$. Let r be the radius of the circle where the sphere intersects a plane parallel to the yz plane, i.e. $r = \sqrt{1 - x^2}$. Then we can set up the surface area integral as

   $$\int_{x=a}^{x=b} 2\pi r \sqrt{dx^2 + dr^2}.$$

The integrand becomes

$$2\pi r\sqrt{dx^2+dr^2} = 2\pi\sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}\,dx = 2\pi\,dx$$

So the integral evaluates to $2\pi(b-a)$, which depends on the distance $b-a$, but not on the values of a and b.

4. **Mashed Potatoes.** A group of n people sit around a table with plates of mashed potatoes in front of them. They all have an equal amount of mashed potatoes except for one person who has slightly more. In order to be more fair, each person puts exactly $\frac{1}{2}$ of their potatoes onto the plate of the person to their right. Then they repeat this process with the potatoes now on their plates, and repeat it again infinitely many times. Prove that each person at the table has more potatoes than anyone else infinitely many times.

Label the people from 0 to $n-1$ mod n. Note that if everyone has the same amount of mashed potatoes on their plate, the potatoes are exchanged evenly at each step. Therefore we may assume that at the beginning, person 0 has one unit of mashed potatoes on their plate, and everyone else has none–this is the same as person 0 having one more unit of potatoes than everyone else. We will show that after t steps, person k has

$$P(t,k) = 2^{-t}\sum_{\substack{0\le j\le t\\ j\equiv k \mod n}}\binom{t}{j}.$$

We can view this process as a lazy random walk (stay with probability 1/2 and move with probability 1/2). There are 2^t different paths of length t that start from 0 and each is equally likely. In order to end up at k, the number of times j that we move (instead of staying) must satisfy $j \equiv k \pmod{n}$. There are $\binom{t}{j}$ ways to choose the j times that we move. So the probability we end up at k is given by the formula above.

We can also prove this formula by induction. This is true when $t=0$, since $P(0,0)=1$ and $P(0,k)=0$ for all $k \neq 0$. For the inductive step, we have

$$\begin{aligned}
P(t+1,k) &= \frac{1}{2}P(t,k-1) + \frac{1}{2}P(t,k)\\
&= 2^{-t-1}(\sum_{\substack{0\le j\le t\\ j\equiv k-1 \mod n}}\binom{t}{j} + \sum_{\substack{0\le j\le t\\ j\equiv k \mod n}}\binom{t}{j}\\
&= 2^{-t-1}\sum_{\substack{0\le j\le t+1\\ j\equiv k \mod n}}\binom{t}{j}+\binom{t}{j-1}\\
&= 2^{-t-1}\sum_{\substack{0\le j\le t+1\\ j\equiv k \mod n}}\binom{t+1}{j}
\end{aligned}$$

which proves the claim.

When $t \pmod{n} \geq \max\{k, n-k\}$, we have $P(t,k) = P(t,n-k)$ by the symmetry of the binomial coefficient. To ease notation, we extend our definition of P for $s \neq n$ by setting $P(t,s) = P(t, s \pmod{n})$, and talk about "person s" instead of "person $s \pmod{n}$".). We will prove by induction that

$$P(t,\lfloor\frac{t}{2}\rfloor) > P(t,\lfloor\frac{t}{2}\rfloor-1) \geq P(t,\lfloor\frac{t}{2}\rfloor-2) \geq \cdots \geq P(t,\lfloor\frac{t}{2}\rfloor-\lfloor\frac{n}{2}\rfloor)$$

Recall that the people are labeled mod n, so this inequality and the symmetry above give an ordering of the amount of mashed potatoes on each person's plate. The inequality is true when $t=0$. For the inductive step, we will use the recursive relation

$$P(t+1,k+1) = \frac{1}{2}P(t,k+1) + \frac{1}{2}P(t,k).$$

when $\lfloor \frac{t}{2} \rfloor - \lfloor \frac{n}{2} \rfloor + 1 \le k \le \lfloor \frac{t}{2} \rfloor - 1$, we may assume inductively that $P(t,k+1) \ge P(t,k) \ge P(t,k-1)$, so

$$P(t+1,k+1) = \frac{1}{2}P(t,k+1) + \frac{1}{2}P(t,k) \ge \frac{1}{2}P(t,k) + \frac{1}{2}P(t,k-1) = P(t+1,k)$$

with the inequality being strict if $k = \lfloor \frac{t}{2} \rfloor - 1$.

For t odd, the bounds on k translate to $\lfloor \frac{t+1}{2} \rfloor - \lfloor \frac{n}{2} \rfloor \le k \le \lfloor \frac{t+1}{2} \rfloor - 2$, so it remains to prove the inequality for $k = \lfloor \frac{t+1}{2} \rfloor - 1$. In that case, $P(t,k+1) = P(t,k)$, so the strict inequality follows by induction as above.

For t even, these bounds on k translate to $\lfloor \frac{t+1}{2} \rfloor - \lfloor \frac{n}{2} \rfloor + 1 \le k \le \lfloor \frac{t+1}{2} \rfloor - 1$, so it remains to prove the inequality for $k = \lfloor \frac{t+1}{2} \rfloor - \lfloor \frac{n}{2} \rfloor$. In that case, $P(t,k-1) = P(t,k+1)$ if n is even and $P(t,k-1) = P(t,k)$ if n is odd. Either way, the desired inequality follows by induction as above.

We now have that for all $t \equiv k \mod n$, after $2t$ steps, person k has more potatoes than anyone else.

We prove that

$$\binom{2n}{0} + \binom{2n}{n} + \binom{2n}{2n} = 2 + \binom{2n}{n} > \binom{2n}{k} + \binom{2n}{n+k} = \binom{2n}{k} + \binom{2n}{n-k} \quad 0 < k < n.$$

First, let us consider $k = n-1$

$$\begin{aligned}
\binom{2n}{n-1} + \binom{2n}{1} &= \frac{2n!}{(n-1)!(n+1)!} + 2n \\
&= \frac{n}{n+1}\binom{2n}{n} + 2n \\
&= \binom{2n}{n} - \frac{1}{n+1}\binom{2n}{n} + 2n = \binom{2n}{n} - C_n + 2n
\end{aligned}$$

where C_n is the nth Catalan number C_n, whose sequence is $1, 2, 5, 14, 42, \ldots$. For $n > 3$, we have $C_n > 2n$. We have $-C_1 + 2 = 1$ and $-C_2 + 4 = 2$ and $-C_3 + 6 = 1$. This confirms that

$$\binom{2n}{n-1} + \binom{2n}{1} \le 2 + \binom{2n}{n}.$$

Next, we show that for $n/2 \le k < n-1$, we have

$$\binom{2n}{k} + \binom{2n}{n-k} \le \binom{2n}{k+1} + \binom{2n}{n-k-1}.$$

This is equivalent to

$$\binom{2n}{k} - \binom{2n}{k+1} \le \binom{2n}{n-k-1} - \binom{2n}{n-k}$$

After dividing by $(2n)!$, the left hand side is

$$\frac{1}{k!(2n-k)!} - \frac{1}{(k+1)!(2n-k-1)!} = \frac{k+1-(2n-k)}{(k+1)!(2n-k)!} = \frac{2k+1-2n}{(k+1)!(2n-k)!} < 0.$$

and the right hand side is

$$\frac{1}{(n-k-1)!(n+k+1)!} - \frac{1}{(n-k)!(n+k)!} = \frac{n-k-(n+k+1)}{(n-k)!(n+k+1)!}$$
$$= \frac{-2k-1}{(n-k)!(n+k+1)!} < 0$$

5. **Closure.** Classify all sets S of positive integers with the following properties:

   - If $x \in S$ then $x + 13 \in S$.
   - If $x \in S$ and x is even then $\frac{x}{2} \in S$.

   The possibilities are $\emptyset$, $\{x \in \mathbb{N} : 13|x\}$, $\{x \in \mathbb{N} : 13 \nmid x\}$, and $\mathbb{N}$. Given any element $x \in S$, we can either divide by 2 or add 13 and divide by 2. This will always produce a smaller element, unless $x \leq 13$. Moreover, if x is a multiple of 13, then this process will always produce multiples of 13; if x is not a multiple of 13 then this process will always produce non-multiples of 13. From this we can see that if any multiple of 13 is in S, then $13 \in S$, and therefore all multiples of 13 are in S. If any non-multiple of 13 is in S, then some positive integer less than 13 is in S. Moreover, by dividing by 2 or adding 13 and dividing by 2, we have

$$1 \in S \Rightarrow 7 \in S \Rightarrow 10 \in S \Rightarrow 5 \in S \Rightarrow 9 \in S \Rightarrow 11 \in S$$
$$\Rightarrow 12 \in S \Rightarrow 6 \in S \Rightarrow 3 \in S \Rightarrow 8 \in S \Rightarrow 4 \in S \Rightarrow 2 \in S \Rightarrow 1 \in S$$

   So as soon as we have one positive integer less than 13, we have all the positive integers less than 13, and hence all non-multiples of 13 in S. Since the two operations on S cannot transform a multiple of 13 into a non-multiple of 13 or vice versa, S is determined by two independent choices: whether or not it contains all the multiples of 13, and whether or not it contains all the non-multiples of 13. Thus S is one of the four sets above.

6. **A Complex Sum.** Find

$$\lim_{n\to\infty} \sum_{k=-n}^{n} \frac{1}{n+ik}.$$

   Here $i = \sqrt{-1}$.

   If we pair off positive and negative terms, we have

$$\sum_{k=-n}^{n} \frac{1}{n+ik} = \frac{1}{n} + \sum_{k=1}^{n} \frac{1}{n+ik} + \frac{1}{n-ik} = \frac{1}{n} + \sum_{k=1}^{n} \frac{2n}{n^2+k^2}$$

   The $\frac{1}{n}$ term vanishes in the limit, and can be ignored. The rest can be rewritten as a Riemann sum:

$$\sum_{k=1}^{n} \frac{2n}{n^2+k^2} = 2\sum_{k=1}^{n} \frac{1}{1+\left(\frac{k}{n}\right)^2} \cdot \frac{1}{n} \to 2\int_0^1 \frac{1}{1+x^2}\,dx = 2\arctan(1) = \frac{\pi}{2}$$

   as $n \to \infty$.

   We can also solve this problem using a Riemann sum for the complex logarithm–one just needs to be careful with how the log is defined on $\mathbb{C}$.

7. **A Real Integral.** Suppose that $f(x)$ is the unique solution to the differential equation

$$f(x) - f'(x) = \sec(x)$$

such that $f(0) = 0$. Find

$$\int_0^{\pi/4} f(x)\sin(x)\,dx$$

The solution to this differential equation is a hypergeometric function, so that approach should be avoided. Set

$$A = \int_0^{\pi/4} f(x)\sin(x)\,dx, \qquad B = \int_0^{\pi/4} f(x)\cos(x)\,dx$$
$$C = \int_0^{\pi/4} f'(x)\sin(x)\,dx, \qquad D = \int_0^{\pi/4} f'(x)\cos(x)\,dx$$

By integration by parts, we have $A - D = -\frac{\sqrt{2}}{2}f(\pi/4)$, $B + C = \frac{\sqrt{2}}{2}f(\pi/4)$. Furthermore, by the differential equation above, we have $A - C = \int_0^{\pi/4} \tan(x)\,dx = \log(2)/2$ and $B - D = \int_0^{\pi/4} 1\,dx = \pi/4$. Solving the linear system, we have $A = \log(2)/4 - \pi/8$.

8. **Nested Shapes.** A regular dodecahedron (12 pentagonal faces, 20 vertices) is inscribed in a regular icosahedron (20 triangular faces, 12 vertices), which is inscribed in a regular dodecahedron. The ratio of side lengths between the inner and outer dodecahedra is A. A regular icosahedron is inscribed in a regular dodecahedron, which is inscribed in a regular icosahedron. The ratio of side lengths between the inner and outer icosahedra is B. Prove one of the following claims: $A > B$, $A = B$, $A < B$. (Note: when one polyhedron is inscribed in another, the vertices of the inner polyhedron lie at the centers of the faces of the outer one.)

   We will prove that $A = B$. Picture a dodecahedron inside an icosahedron inside a dodecahedron inside an icosahedron, all centered at the origin. Each center of a face of the inner dodecahedron lies on the line segment from the origin to a vertex of the inner icosahedron. This vertex is the center of a face of the outer dodecahedron. Thus, if we rescale from the origin by a factor of A, we will transform the inner dodecahedron into the outer one. This same rescaling maps the center of each face of the inner icosahedron (which is a vertex of the inner dodecahedron) to a center of a face of the outer icosahedron (which is a vertex of the outer dodecahedron). Therefore this rescaling transforms the inner icosahedron to the outer one, so it is also a rescaling by a factor of B. We conclude that $A = B$.

9. **Fix this Problem.** The following problem was intended to be on the 2020 Konhauser, but when we received it, the ink was smudged on a crucial piece of information.

   "Find positive integers a, b, c, d, e, f such that

$$\text{lcm}(a,b,c) = 60, \quad \text{lcm}(b,c,d) = 540, \quad \text{lcm}(c,d,e) = 135,$$
$$\text{lcm}(d,e,f) = 5454, \quad \text{lcm}(e,f,a) = 1212, \quad \text{lcm}(f,a,b) = \text{(illegible)}.$$

   Here lcm denotes the least common multiple of a set of integers."

   The problem was intended to have a unique solution consisting of distinct positive integers a, b, c, d, e, f. Find the illegible number, and find (a,b,c,d,e,f).

   Based on this information, the possible prime factors of a,b,c,d,e,f are $2, 3, 5, 101$. It is helpful to work with each prime separately. Let n_p denote the number of times p divides n. We have

$$\max(a_2, b_2, c_2) = 2, \quad \max(b_2, c_2, d_2) = 2, \quad \max(c_2, d_2, e_2) = 0,$$
$$\max(d_2, e_2, f_2) = 1, \quad \max(e_2, f_2, a_2) = 2, \quad \max(f_2, a_2, b_2) = x_2$$

   Therefore $a_2 = b_2 = 2$, $c_2 = d_2 = e_2 = 0$ $f_2 = 1$, and x_2 must be 2.

We have

$$\max(a_3, b_3, c_3) = 1, \quad \max(b_3, c_3, d_3) = 3, \quad \max(c_3, d_3, e_3) = 3,$$
$$\max(d_3, e_3, f_3) = 3, \quad \max(e_3, f_3, a_3) = 1, \quad \max(f_3, a_3, b_3) = x_3$$

Therefore $d_3 = 3$ and $a_3, b_3, c_3, e_3, f_3 \le 1$. The only way to have a unique solution is with $x_3 = 0$, which makes $a_3 = b_3 = f_3 = 0$, $c_3 = e_3 = 1$.

We have

$$\max(a_5, b_5, c_5) = 1, \quad \max(b_5, c_5, d_5) = 1, \quad \max(c_5, d_5, e_5) = 1,$$
$$\max(d_5, e_5, f_5) = 0, \quad \max(e_5, f_5, a_5) = 0, \quad \max(f_5, a_5, b_5) = x_5$$

Therefore $a_5 = d_5 = e_5 = f_5 = 0$, $c_5 = 1$. We have a unique solution with $x_5 = 0$, which makes $b_5 = 0$, or $x_5 = 1$, which makes $b_5 = 1$. However, notice that $a_2 = b_2$, $a_3 = b_3$, and (we will see) $a_{101} = b_{101}$. Since we want $a \neq b$, we must take $x_5 = b_5 = 1$.

We have

$$\max(a_{101}, b_{101}, c_{101}) = 0, \quad \max(b_{101}, c_{101}, d_{101}) = 0, \quad \max(c_{101}, d_{101}, e_{101}) = 0,$$
$$\max(d_{101}, e_{101}, f_{101}) = 1, \quad \max(e_{101}, f_{101}, a_{101}) = 1, \quad \max(f_{101}, a_{101}, b_{101}) = x_{101}$$

Therefore $a_{101} = b_{101} = c_{101} = d_{101} = e_{101} = 0$, $f_{101} = 1$, and x_{101} must be 1.

Combining all this, we have $x = 2020$, $(a, b, c, d, e, f) = (4, 20, 15, 27, 3, 202)$.

10. **A Curious Matrix.** The function $\cos(\frac{2\pi}{n}x)$ is applied to the entries of an $n \times n$ multiplication table. The 3×3 example is shown below:

$$\begin{pmatrix} \cos(\frac{2\pi}{3} \cdot 1) & \cos(\frac{2\pi}{3} \cdot 2) & \cos(\frac{2\pi}{3} \cdot 3) \\ \cos(\frac{2\pi}{3} \cdot 2) & \cos(\frac{2\pi}{3} \cdot 4) & \cos(\frac{2\pi}{3} \cdot 6) \\ \cos(\frac{2\pi}{3} \cdot 3) & \cos(\frac{2\pi}{3} \cdot 6) & \cos(\frac{2\pi}{3} \cdot 9) \end{pmatrix}$$

Show that the any eigenvalue of this $n \times n$ matrix is either 0 or $\pm\sqrt{n}$.

Call this matrix A. We will compute A^2 and A^3. The i, j entry of A^2 has the form

$$\begin{aligned} \sum_{k=1}^{n} \cos\left(\frac{2\pi}{n} ik\right) \cos\left(\frac{2\pi}{n} jk\right) &= \sum_{k=1}^{n} \cos\left(\frac{2\pi}{n} \frac{(i+j)}{2} k\right) + \cos\left(\frac{2\pi}{n} \frac{(i-j)}{2} k\right) \\ &= \sum_{k=1}^{n} \cos\left(\frac{\pi}{n}(i+j)k\right) + \cos\left(\frac{\pi}{n}(i-j)k\right). \end{aligned}$$

This sum vanishes unless $i + j = n$, $i + j = 2n$, or $i - j = 0$. It has the value $n/2$ if one of these conditions holds, and n if two hold. Therefore $A^2 = n/2 \cdot (I + J)$, where I is the identity matrix and J is the matrix with 1's in positions $i, n - i - 1$ and n, n, and 0's elsewhere.

Note that the ith row of A and the $n - 1 - i$th row are identical, because $\cos(\frac{2\pi}{n} ij) = \cos(\frac{2\pi}{n}(n - i)j)$. Therefore $JA = A$, and we find that $A^3 = nA$. Any eigenvalue λ of A must satisfy $\lambda^3 = n\lambda$, so $\lambda = 0$ of $\lambda = \pm\sqrt{n}$.